UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Complex Analysis

August 14, 2025

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$

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1. What is the group of all biholomorphisms of $\mathbb{D} \setminus \{0\}$? Justify your answer.

2. Show that there is no proper holomorphic map $f : \mathbb{D} \to \mathbb{C}$. Recall that "proper" means that $|z_n| \to 1$ implies $|f(z_n)| \to \infty$ for any sequence $z_n \in \mathbb{D}$.

3. Using residue calculus evaluate

$$\int_0^\infty \frac{1}{x^3+1} dx$$

4. Consider the upper half-plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$$

equipped with the hyperbolic metric. Show that the hyperbolic distance from $n + in$ to the positive imaginary axis does not depend on $n \in \{1, 2, 3, \cdots\}$.

5. Let $f_1, f_2, ... : \mathbb{C} \to \mathbb{C}$ be a sequence of non-vanishing entire functions that converge uniformly on compact sets to an entire function f_∞. Show that f_∞ is either identically 0 or non-vanishing.

6. Let $A_R = \{z \in \mathbb{C} \mid 1 < |z| < R\}$ for $R > 1$. If A_R and $A_{R'}$ are biholomorphic, show that $R = R'$.