Name:

WRITTEN QUALIFYING EXAM
MATH 6310/SUMMER 2025

Answer all five of the questions. Show all your work.

Three completely correct answers = high pass = A

Two and a half correct answers = pass = A-

1. Let p be a prime number.

(a) Find all the abelian groups with p^6 elements.

(b) How many abelian groups are there with a million (i.e. 10^6) elements?

2. Find all the maximal ideals in the following rings:

(a)

$$\mathbb{Q}[x,y]/\langle xy-2, x^2-y^2\rangle$$

(b)

$$\mathbb{R}[x,y]/\langle xy-2, x^2-y^2\rangle$$

(c)

$$\mathbb{C}[x,y]/\langle xy-2, x^2-y^2\rangle$$

3. Suppose that A is an $n \times n$ matrix with characteristic polynomial

$$(x-1)^n$$

Prove that if $A^d = A^e$ for any pair $d \neq e$ of integers, then $A = I_n$.

And no, you cannot just quote the theorem that a matrix of finite order is diagonalizable. We want you to demonstrate that you know why it is true.

4. Consider the maximal ideal $\langle x, y\rangle \subset k[x, y]$ for a field k.

(a) Prove that the kernel of the natural map of $k[x, y]$-modules:

$$\langle x, y\rangle \otimes_{k[x,y]} \langle x, y\rangle \to \langle x, y\rangle^2$$

is a one-dimensional vector space over k and find a generator of it.

(b) Find the other two Tors:

$$\mathrm{Tor}_1^{k[x,y]}(\langle x, y\rangle, \langle x, y\rangle) \text{ and } \mathrm{Tor}_2^{k[x,y]}(\langle x, y\rangle, \langle x, y\rangle)$$

5. What about the Homomorphisms in Problem 4? Namely,

(a) Is it true that:

$$\mathrm{Hom}_{k[x,y]}(\langle x, y\rangle, \langle x, y\rangle) = k \cdot \mathrm{id}$$

is a one-dimensional vector space, or are there more $k[x, y]$-module homomorphisms than just the multiples of the identity?

(b) Are there any non-split extensions of $k[x, y]$-modules:

$$0 \to \langle x, y\rangle \to M \to \langle x, y\rangle \to 0$$

i.e. is

$$\mathrm{Ext}^1_{k[x,y]}(\langle x, y\rangle, \langle x, y\rangle) = 0 \text{ or not?}$$