UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Partial Differential Equations
August 14/15, 2025.

Instructions: There are six total problems. Problems will be scored out of 10 points and four (4) problems will be graded. Clearly identify which four (4) problems you want to be graded. Partial credit will be given for *significant* progress towards the solution.

Three (3) completely correct problems will be a High Pass and two (2) completely correct problems with sufficient partial credit for at least 26 total points will be a Pass. Be sure to provide all relevant definitions and statements of theorems cited. All solutions must include rigorous justification unless otherwise indicated.

Problem 1. Let U be a bounded domain in $\mathbb{R}^d$ and Γ be a closed subset of ∂U. Consider the energy

$$I[v] = \frac{1}{2}\int_U |Dv|^2 + u^2 \ dx$$

on the admissible class

$$\mathcal{A} = \{v \in C^2(\overline{U}) \cap C^1(\overline{U}) : u|_\Gamma = g\}, \ g \in C(\partial U).$$

Suppose that u is a minimizer for the calculus of variations problem

$$I[u] = \min_{v \in \mathcal{A}} I[v]$$

find and justify the PDE boundary value problem solved by u. Show that there is at most one solution of this PDE in the admissible class $\mathcal{A}$.

Problem 2. Let $c > 0$ and $\mu(x)$ a smooth function. Consider the wave equation

$$\begin{cases} \partial_t^2 u - c^2 \Delta u + \mu(x)\partial_t u = f & \text{in } (0,T] \times \mathbb{R}^d \\ (u, u_t) = (\phi, \psi) & \text{on } \{t = 0\} \times \mathbb{R}^d \end{cases} \tag{1}$$

Explain the notion of *domain of dependence* in the context of wave-type PDEs. Show finite speed of propagation for this PDE by showing, using an energy method, a natural upper bound on the domain of dependence of some point (t_0, x_0).

Problem 3. Let U be a bounded domain and u solve the heat equation

$$\begin{cases} \partial_t u = \Delta u + cu & \text{in } U \times (0,\infty) \\ u(x,0) = g(x) & \\ u(x,t) = 0 & \text{on } \partial U \times (0,\infty). \end{cases}$$

Classify, with justification, the optimal range of values for $c \in \mathbb{R}$ so that for any smooth initial data g the solution u above converges to 0 as $t \to \infty$.

Problem 4. Find the entropy solution of the following scalar conservation law, arising in traffic flow modeling,

$$\begin{cases} \partial_t u + \partial_x(u(1-u)) = 0 & \text{for } x \in \mathbb{R},\ t > 0 \\ u(x,0) = g(x) \end{cases}$$

with initial data

$$g(x) = \begin{cases} 0 & x < -1 \\ 1 & -1 < x < 0 \\ 0 & 0 < x < 1 \\ 1/2 & 1 < x. \end{cases}$$

After the rarefaction hits the shock you should write the explicit ODE IVP for the shock location $\gamma(t)$ and a piecewise definition of the solution using this shock location $\gamma(t)$, but you don't actually need to solve the ODE IVP to get the explicit formula for $\gamma(t)$.

Problem 5. Let $f : \mathbb{R}^d \to \mathbb{R}$ smooth. Show that there is *at most one* smooth solution of

$$\begin{cases} -\Delta u = f & \text{in } \mathbb{R}^d \\ \lim_{|x| \to \infty} u(x) = 0. \end{cases}$$

Give a (non-trivializing) additional condition on f so that you can show existence of such a solution.

Problem 6. Suppose that $U \subset \mathbb{R}^n$ is a bounded domain and $b : U \to \mathbb{R}^n$ is a smooth velocity field. Suppose that $u(x,t)$ solves the reaction-diffusion-advection equation with a logistic reaction term

$$\begin{cases} \partial_t u + b(x) \cdot \nabla u - \Delta u = u(1-u) & \text{in } U \times (0,\infty) \\ u = 0 & \text{on } \partial U \times [0,\infty). \end{cases}$$

Show that if $u(x,0) < 1$ in U then $u(x,t) < 1$ for all $t > 0$ and all $x \in U$.