Preliminary Exam, Math 6610 August 2025

Instructions: This exam is closed book, no notes, and no electronic devices are allowed. You have two hours and you need to work on any 2 of problems (1,2,3) and either 1 of problems (4,5). All questions have equal weight and a score of 65% is considered a pass, and a score of 80% is considered a high pass. Indicate clearly the work that you wish to be graded.

1) Least Squares Problems. Let $A \in \mathbb{R}^{mxn}$ with $m > n$ have rank n and $\mathbf{b} \in \mathbb{R}^m$. The Least Squares problem is to find $\mathbf{x} \in \mathbb{R}^n$ which makes $\|A\mathbf{x} - \mathbf{b}\|_2$ as small as possible.

a) Show that $\mathbf{x}$ solves this Least Squares problem if and only if $A^T A\mathbf{x} = A^T\mathbf{b}$.

b) What is meant by the QR factorization of matrix A?

c) Show how to use the QR factorization of A to solve the Least Squares problem.

2) Sensitivity. For nonsingular $A \in \mathbb{R}^{n\times n}$, and nonzero $\mathbf{b} \in \mathbb{R}^n$, consider the problems

$$A\mathbf{x} = \mathbf{b},$$
$$\hat{A}\hat{\mathbf{x}} = \mathbf{b}.$$

a) Show that

$$\frac{\|\mathbf{x} - \hat{\mathbf{x}}\|}{\|\mathbf{x}\|} \leq \|A\|\|A^{-1}\|\frac{\|A - \hat{A}\|}{\|A\|}$$

b) Give a bound on $\|A - \hat{A}\|_2$ which ensures that $\hat{A}$ is nonsingular if A is nonsingular.

3) Singular Value Decomposition. Let $A \in \mathbb{R}^{m\times n}$, with $m \geq n$.

a) What is the singular value decomposition (SVD) of A? Does every such matrix A have an SVD? Prove that it does or show an example of a matrix A which does not have an SVD.

b) For any $m \times n$ matrix A with an SVD and $\epsilon > 0$, show that there is a full-rank matrix B for which $\|A - B\|_2 < \epsilon$.

c) For $m \times n$ matrix A, give orthonormal bases for the subspaces range(A), null(A), range(A*), and null(A*).

4). Solution of a nonlinear fixed point problem

Consider the fixed-point problem $x = g(x)$ for a continuously differentiable real valued function g and assume that it has a solution α for which $|g'(\alpha)| < 1$. State and prove a convergence theorem for fixed point iteration $x^{(k+1)} = g(x^{(k)})$ for this problem. **Note: Do not assume more about the function g than is stated in the problem.**

5) Fixed-point iteration for linear systems.

(a) For $T \in \mathbb{R}^{m\times m}$ and any matrix norm, show that $\|T^k\| \to 0$ as $k \to \infty$ if and only if $\rho(T) < 1$ where $\rho(T)$ is the spectral radius of T. Do **not** assume that the matrix T is diagonalizable.

(b) Suppose $A \in \mathbb{R}^{m\times m}$ is nonsingular, that $\mathbf{b} \in \mathbb{R}^m$, and that $A\mathbf{x} = \mathbf{b}$. Suppose that $\mathbf{x} = T\mathbf{x}+\mathbf{c}$ is a fixed point problem equivalent to this linear system. What are the implications of part (a) for the convergence of the fixed-point iteration $\mathbf{x}^{(k+1)} = T\mathbf{x}^{(k)} + \mathbf{c}$ to the solution $\mathbf{x}$ of the linear system?

(c) Suppose A is a symmetric, weakly diagonally dominant, and irreducible matrix. Define the Jacobi iterative method for solving $A\mathbf{x} = \mathbf{b}$, and show that this method converges to the solution of the linear system starting with any initial vector $\mathbf{x}^{(0)}$. (*Hint: You might find it helpful to use the fact that a matrix which is both weakly diagonally dominant and irreducible is nonsingular.*)

Recall that: An $m\times m$ matrix A is <u>irreducible</u> iff for all $i, j \in \{1, 2, ..., m\}$, $i \neq j$, either $a_{ij} \neq 0$ or there are i_1, i_2, ..., i_s, each between 1 and m, such that $a_{i,i_1}a_{i_1,i_2}a_{i_2,i_3}\cdots a_{i_s,j} \neq 0$. A symmetric $m \times m$ matrix A is <u>weakly diagonally dominant</u> if $|a_{ii}| \geq \sum_{j\neq i} |a_{ij}|$ for all i, and $|a_{ii}| > \sum_{j\neq i} |a_{ij}|$ for at least one i.