Department of Mathematics, University of Utah
Ph.D. Preliminary Examination: Analysis of Numerical Methods, II
Fall 2025

This exam is closed book, closed notes, and no calculators are allowed. You have two hours to complete this exam.

There are 5 problems below. **You must complete 3 problems total.**

- **You must complete 2 problems from questions 1-3.**
- **You must complete 1 problem from questions 4-5.**

Each problem is worth 20 points. Clearly indicate which 3 problems you wish to be graded; otherwise, the first 3 problems with work shown will be graded.

- A score of 52 (out of 60) is a *high pass*.
- A score of 48 (out of 60) is a *pass*.

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1. (20 pts) On an equidistant mesh $x_j = jh$, define the operator $\widetilde{D}_0$ as,

$$\widetilde{D}_0 u(x_j) = \frac{u(x_j + h/2) - u(x_j - h/2)}{h},$$

For a given smoooth κ and f, consider the scheme,

$$-\widetilde{D}_0 \left(\kappa(x_j) \widetilde{D}_0 u_j \right) = f(x_j),$$

for the ODE $-\frac{d}{dx}\left(\kappa(x)\frac{d}{dx}u(x)\right) = f(x)$. What order is the local truncation error?

2. (20 pts) For the ODE $\boldsymbol{u}'(t) = \boldsymbol{f}(t, \boldsymbol{u})$, consider the time-integration scheme

$$\boldsymbol{u}_{n+1} = \boldsymbol{u}_n + \frac{k}{4}\boldsymbol{f}(t_n, \boldsymbol{u}_n) + \frac{3k}{4}\boldsymbol{f}\left(t_n + \frac{2}{3}k, \boldsymbol{u}_n + \frac{2}{3}k\boldsymbol{f}(t_n, \boldsymbol{u}_n)\right).$$

(a) (5 pts) Identify the Butcher tableau for this scheme.
(b) (8 pts) Compute the order of consistency of this scheme.
(c) (7 pts) Compute the stability/amplification factor of the scheme.

3. (20 pts) Consider the ODE,

$$\boldsymbol{u}'(t) = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{N}(t, \boldsymbol{u}),$$

where $\boldsymbol{A}$ is a fixed matrix and $\boldsymbol{N}$ is an arbitrary, e.g., nonlinear, function.

(a) (6 pts) With initial data $\boldsymbol{u}(0) = \boldsymbol{u}_0$, show that the solution to this IVP at time $t > 0$ is given by,

$$\boldsymbol{u}(t) = e^{t\boldsymbol{A}}\boldsymbol{u}_0 + \int_0^t e^{(t-s)\boldsymbol{A}}\boldsymbol{N}(s, \boldsymbol{u}(s))\mathrm{d}s,$$

where $e^{t\boldsymbol{A}}$ is the matrix exponential of $t\boldsymbol{A}$.

(b) (14 pts) Exponential integrators form a scheme to compute $\boldsymbol{u}^{n+1}$ in terms of $\boldsymbol{u}^n$ by integrating the above expression over the interval $[t_n, t_{n+1}]$ (instead of $[0, t]$), by approximating $\boldsymbol{N}(\boldsymbol{u}(s))$ with a quadrature rule/polynomial approximation, and by exactly integrating the matrix exponential term. A Forward Euler exponential integrator makes the approximation $\boldsymbol{N}(\boldsymbol{u}(s)) \approx \boldsymbol{N}(\boldsymbol{u}_n)$. Write out the Forward Euler exponential integrator scheme explicitly (i.e., by eliminating integrals).

4. (20 pts) For the ODE $u'(t) = f(t,u)$, consider the scheme,

$$u_{n+1} + \alpha_1 u_n = k\beta_0 f_{n+1} + k\beta_1 f_n,$$

where $u_n \approx u(t_n)$, $f_n = f(t_n, u_n)$, $k = t_{n+1} - t_n$, and $u' = f(t,u)$.

(a) (10 pts) Identify the coefficients α, β that yield a scheme of optimal k-order of accuracy, and identify this order of accuracy.

(b) (10 pts) Determine whether or not this scheme is 0-stable and/or A-stable.

5. (20 pts)

(a) (10 pts) Use von Neumann stability analysis to determine a stability condition for,

$$D^+ u_j^n = D_0 u_j^n,$$

where $D^+ u_j^n = (u_j^{n+1} - u_j^n)/k$, $D_0 u_j^n = (u_{j+1}^n - u_{j-1}^n)/(2h)$, $D_\pm u_j^n = \pm(u_{j\pm1}^n - u_j^n)/h$.

(b) (10 pts) Use von Neumann stability analysis to determine a stability condition for

$$D^+ u_j^n = D_+ D_- u_j^n.$$