UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Real Analysis
Jan 6, 2022.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

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1. Prove the Borel-Cantelli lemma: Let $(X, \mathcal{M}, \mu)$ be a measure space and let A_i be measurable sets such that $\sum_{i=1}^{\infty} \mu(A_i) < \infty$. Then the set

   $$\Omega = \{x \in X \mid x \in A_i \text{ for infinitely many } i\}$$

   of points that belong to infinitely many A_i has measure 0.

2. Let X be a compact metric space, $\mathcal{M}$ the Borel σ-algebra on X, and μ a finite measure on $\mathcal{M}$ (meaning that $\mu(X) < \infty$; we call such a measure a *finite Borel measure* on X). Prove that μ is *regular*, i.e. for every Borel set $E \subset X$

   - $\mu(E) = \sup\{\mu(K) \mid K \subseteq E \text{ is compact}\}$, and
   - $\mu(E) = \inf\{\mu(U) \mid U \supseteq E \text{ is open}\}$.

   Hint: Show that the set of E's that satisfy both bullets is a σ-algebra that contains compact sets.

3. (a) Let $(X, \mathcal{M})$ be a measurable space, μ, ν positive measures on $(X, \mathcal{M})$. Suppose $\nu(X) < \infty$. Prove that $\nu \ll \mu$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu(E) < \epsilon$.

   (b) For μ the Lebesgue measure on $\mathbb{R}$ find a positive (but infinite) measure ν such that $\nu \ll \mu$ but the $\epsilon - \delta$ statement in (a) fails.

4. Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $f : X \to X$ be a measure preserving bijection, in the sense that $\mu(A) = \mu(f(A))$ for every $A \in \mathcal{M}$. Prove that for every $E \in \mathcal{M}$

   $$\{x \in E \mid f^n(x) \notin E \text{ for all } n > 0\}$$

   has measure 0.

5. Let V, W be Banach spaces, $T_i : V \to W$ a sequence of bounded linear operators such that $\lim_{i\to\infty} Tv$ exists for every $v \in V$. Define the linear map $T : V \to W$ by
$$Tv = \lim_{i\to\infty} Tv$$
Show that T is bounded.

6. Show that the Banach space ℓ^p with the usual norm is not a Hilbert space when $p \in [1, 2) \cup (2, \infty)$.