UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Real Analysis
January, 2024.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

Let λ denote Lebesgue measure on $\mathbb{R}$.

1. Let X be a set and μ^* be an outer measure on X. A set $A \subset X$ is called μ^*-measurable if for every $S \subset X$ we have
$$\mu^*(S) = \mu^*(S \cap A) + \mu^*(S \setminus A).$$
Show that μ^* is countably additive on the μ^* measurable sets.

2. Prove that there exists $\phi : \ell^2(\mathbb{N}) \to \mathbb{R}$ that is linear and not continuous.

3. Show that the set of real numbers that have infinitely many 2s in their base 10 expansion is a Borel set.

4. Prove that if $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is continuous and linear and $AB(0,1)$ is dense in $B(0,1)$ then A is surjective. Note that this is true for Banach spaces as well (though we are not asking you to prove it).

5. We say $f_1, ...$ is λ-Cauchy if for every $\epsilon > 0$ there exists N so that for $i, j > N$ we have
$$\lambda(\{x : |f_i(x) - f_j(x)| > \epsilon\}) < \epsilon.$$
Show that if $f_1, ... \in L^1(\lambda)$ is λ-Cauchy then there exists $g \in L^1(\lambda)$ so that $f_1, f_2, ...$ converges in measure to g.

6. Let $(X, \mathcal{M})$ be a measurable space and μ and ν be two finite measures on $(X, \mathcal{M})$. Show that the following are equivalent
   - $\nu \ll \mu$
   - Whenever $A_1, ... \in \mathcal{M}$ satisfy $\lim_{n\to\infty} \mu(A_n) = 0$ we have $\lim_{n\to\infty} \nu(A_n) = 0$.