UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Partial Differential Equations
May 30, 2024.

Instructions: There are six total problems. Problems will be scored out of 10 points and four (4) problems will be graded. Clearly identify which four (4) problems you want to be graded. Partial credit will be given for *significant* progress towards the solution.

Three (3) completely correct problems will be a High Pass and two (2) completely correct problems with sufficient partial credit for at least 26 total points will be a Pass. Be sure to provide all relevant definitions and statements of theorems cited. All solutions must include rigorous justification unless otherwise indicated.

Problem 1. Show that $u(x) = \frac{1}{4\pi|x|}$ is the fundamental solution of the Laplace equation in $\mathbb{R}^3$, $-\Delta u = \delta_0$ in the sense of distributions, i.e.

$$\int_{\mathbb{R}^3} u(x)\Delta\varphi(x)\ dx = -\varphi(0)$$

for all φ smooth and compactly supported. You should provide a fully detailed proof, not just a formal computation.

Problem 2. State (carefully) and prove the standard L^2 Poincaré inequality in a smooth bounded domain U.

Problem 3. Let $b(x)$ a smooth vector field on a bounded domain U in $\mathbb{R}^n$. Consider a smooth scalar solution u of the equation

$$\begin{cases} u_t = \Delta u + \nabla\cdot(b(x)u) & \text{in } U\times(0,\infty), \\ u = 0 & \text{on } \partial U\times[0,\infty), \\ u(x,0) = f(x). \end{cases}$$

(a) Show that if $\nabla\cdot b = 0$ then for all $t > 0$

$$\sup_{x\in U} u(x,t) \le \sup_U |f|.$$

(b) Show that if $\nabla\cdot b \neq 0$ then for all $t > 0$

$$\sup_{x\in U} u(x,t) \le e^{At}\sup_U |f| \quad \text{with } A = \sup_U \nabla\cdot b.$$

Problem 4. Let $c, \mu > 0$ Consider the wave equation with nonlinear frictional dissipation

$$\begin{cases} u_{tt} - c^2 \Delta u + \mu u_t^3 = f & \text{in } (0,T] \times \mathbb{R}^d \\ (u, u_t) = (\phi, \psi) & \text{on } \{t = 0\} \times \mathbb{R}^d \end{cases} \quad \text{in} \quad (0,T] \times \mathbb{R}^d \tag{1}$$

Explain the notion of *domain of dependence* in the context of wave-type PDEs. Show finite speed of propagation for this PDE by showing, using an energy method, a natural upper bound on the domain of dependence of some point (t_0, x_0). **Hint:** You do not need to concoct a complicated energy to get perfect energy conservation, just use a simple and standard energy where you can prove energy dissipation (i.e. monotone decreasing).

Problem 5. Let U be a bounded domain. Consider the energy

$$I[v] := \int_U \frac{1}{2} |\nabla v|^2 \, dx$$

on the admissible class

$$\mathcal{A} = \{ v \in C^2(U) \cap C^1(\overline{U}) : \ v = 0 \ \text{ on } \ \partial U, \int_U v \, dx = 1 \}.$$

Suppose that there is $u \in \mathcal{A}$ satisfying

$$I[u] = \min_{v \in \mathcal{A}} I[v].$$

Show that u must solve the PDE boundary value problem

$$\begin{cases} -\Delta u = \lambda & \text{in } U \\ u = 0 & \text{on } \partial U. \end{cases}$$

Here $\lambda \in \mathbb{R}$ is a Lagrange multiplier associated with the constraint. Find a formula for λ in terms of $I[u]$.

Problem 6. Consider the scalar conservation law,

$$u_t + (\tfrac{1}{2} u^2)_x = 0 \ \text{ in } \ \mathbb{R} \times (0, \infty).$$

Find the entropy satisfying weak solution for the initial data,

$$u(x,0) = \begin{cases} 0 & x < 0 \text{ or } x > 1 \\ 1 - x & 0 < x < 1 \end{cases}$$

Make a drawing of the characteristics in the (x,t)-plane including any shocks and rarefactions.