2.5 Finding the zeros of polynomial functions

We will learn how to:

- Determine the number of zeros of polynomial functions
- Find rational zeros of polynomial functions
- Find conjugate pairs of complex zeros
- Find zeros of polynomials by factoring
- Write a polynomial function given the roots.

$$P(x) = a_1x^n + a_2x^{n-1} + . . . +a_{n-1}x + a_n$$

Factored form $a(x - r_1)(x - r_2) \dots (x - r_n)$

The roots are $r_1, r_2, ..., r_n$

Rational Root Theorem:

x-int
zeros
roots

$5/2 \quad -2/3 \quad +1/1$

✓ $P(x) = (2x-5)(3x+2)(x-1) = 0$ Has roots: $x = 5/2, -\frac{2}{3}, 1$

✓ $P(x) = 6x^3 - 17x^2 + x + 10$

set of ps: $\pm\{1, 2, 5, 10\}$ ✓

set of qs: $\pm\{1, 2, 3, 6\}$ ✓

P(x) must have integer coefficients.

Possible: $\pm\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{6}, 2, \frac{2}{3}, 5, \frac{5}{2}, \frac{5}{3}, \frac{5}{6}, 10, \frac{10}{3}\}$

If P(x) has any rational roots, they will be of the form: p/q where p and q have no common factors other than 1 and where p is a factor of the constant term and q is a factor of the leading coefficient.

This allows us to attempt to break higher degree polynomials down into their factored form and determine the roots of a polynomial.

Example 1: Factor completely and determine the roots of this polynomial.

± 1 $\pm\{1, 2\}$

$P(x) = x^3 + 3x^2 + x - 2$

1) set of ps $\pm 1, \pm 2$

2) set of qs ± 1

3) possible roots of P(x) $\pm 1, \pm 2$

4) Test each possible root using synthetic division:

$$\begin{array}{r|rrrr} 1 & 1 & 3 & 1 & -2 \\ & & 1 & 4 & 5 \\ \hline & 1 & 4 & 5 & 3 \end{array} \quad \text{not}$$

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 1 & -2 \\ & & -1 & -2 & 1 \\ \hline & 1 & 2 & -1 & -1 \end{array} \quad \text{not}$$

$$\begin{array}{r|rrrr} 2 & 1 & 3 & 1 & -2 \\ & & 2 & 10 & 22 \\ \hline & 1 & 5 & 11 & \text{Not} \end{array}$$

$$\begin{array}{r|rrrr} -2 & 1 & 3 & 1 & -2 \\ & & -2 & -2 & 2 \\ \hline & 1x^2 & 1x & -1 & \end{array}$$

root = -2

factor (x+2)

$x^3 + 3x^2 + x - 2 = (x+2)(x^2 + x - 1)$ factored form

$x = \dfrac{-1 \pm \sqrt{1+4}}{2 \cdot 1} = \dfrac{-1 \pm \sqrt{5}}{2}$

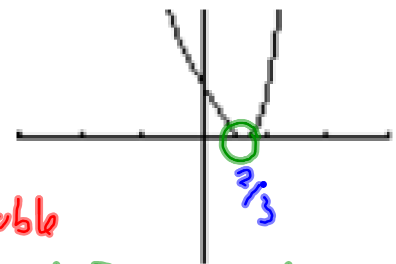
Roots: $-2, \dfrac{-1+\sqrt{5}}{2}, \dfrac{-1-\sqrt{5}}{2}$

Example 2: Find the roots and write in factored form:

$1, 3, 9$ $\qquad \pm 1, 2, 4$

$y = 9x^4 - 3x^3 + x^2 - 8x + 4$

$\{1, 2, 4, \frac{1}{3}, \frac{2}{3}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}\}$



$$\begin{array}{c|ccccc} \frac{2}{3} & 9_{x^4} & -3 & 1 & -8 & 4 \\ & & 6 & 2 & 2 & -4 \\ \hline \frac{2}{3} & 9_{x^3} & 3 & 3 & -6 & ☺ \\ & & 6 & 6 & 6 & \\ \hline & 9_{x^2} & 9 & 9 & ☺ & \end{array}$$

Double Root $2/3 \rightarrow (x-\frac{2}{3})^2$

$(x-\frac{2}{3})(x-\frac{2}{3})(9x^2+9x+9)$

$(x-\frac{2}{3})(x-\frac{2}{3})(3)(3)(x^2+x+1)$

$\underline{(3x-2)(3x-2)(x^2+x+1)}$

Factored form

Roots: $2/3$ (double)

$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{3}}{2} =$

✓ $-\frac{1}{2} + \frac{\sqrt{3}i}{2}$

✓ $-\frac{1}{2} - \frac{\sqrt{3}i}{2}$

✓✓ $2/3$ (double root)

Example 3:

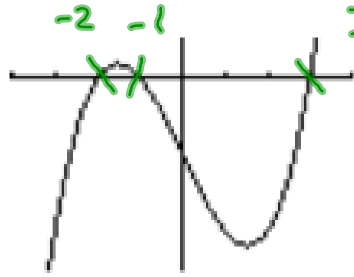
Determine the roots and write in factored form:

x^2 ? $\pm 1, 2, 3, 6$

$y = x^3 - 7x - \underline{6}$

Possible $\pm \{1, 2, 3, 6\}$

$$\begin{array}{r|rrrr} 3 & 1 & 0 & -7 & -6 \\ & & 3 & 9 & 6 \\ \hline & 1 & 3 & 2 & 0 \end{array}$$



$(x-3)(x^2+3x+2) = (x-3)(x+1)(x+2)$

Factored form

$x = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm \sqrt{1}}{2}$

$\frac{-3+1}{2} = -1$ ✓

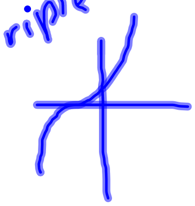
$\frac{-3-1}{2} = -2$ ✓

$x = 3$ ✓

Notice: as soon as you can get the factored form down to a quadratic, use the quadratic formula to find the other two roots. They may be complex.
Complex roots will come in conjugate pairs. If a + bi is a root, then a - bi will be a root if the polynomial has integer coefficients.

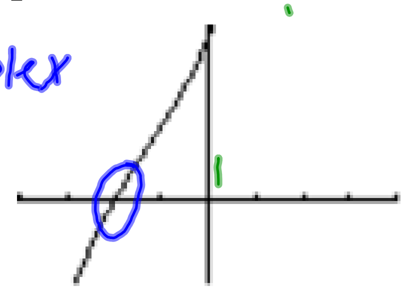
Example 4: Factor and determine the roots:

Triple root



$y = x^3 + 4x^2 + 14x + 20$

1 real -
2 complex



Factored form
$(x+2)(x^2+2x+10)$
Roots:
-2
$-1+6i$
$-1-6i$

$x^2+2x+10=0$

$x = \dfrac{-2 \pm \sqrt{4-40}}{2} = \dfrac{-2 \pm \sqrt{-36}}{2}$

$= \dfrac{-2 \pm 6i}{2}$

$= -1+6i$

$-1-6i$

Example 5:

Write a polynomial function with real coefficients of degree 4 which has these roots:

2i, -3, 1

$0-2i \rightarrow \text{conj} \rightarrow 0+2i$

$(x-2i)(x+2i) = x^2 - 4i^2$

x^2+4

$(x^2+4)(x+3)(x-1)$

$(x^2+4)(x+3) = x^3+3x^2+4x+12$

$\cdot\,(x-1)$

$$
\begin{array}{r}
x^4+3x^3+4x^2+12x \\
-x^3-3x^2-4x-12 \\
\hline
x^4+2x^3+x^2+8x-12
\end{array}
$$