CHAPTER 9: SEQUENCES AND SERIES

9.1 Sequences and Series

In section 9.1 you will learn to:

- Use sequence notation to write the terms of a sequence.
- Use factorial notation.
- Use summation notation to write sums.
- Find the sums of infinite series.
- Use sequences and series to model and solve real-life problems.

1, 3, 5, 7

4!

$$\sum_{i=1}^{5} 2i$$

$$\sum_{k=2}^{\infty} \frac{1}{k^2}$$

What is a sequence?

Finite: 1, 2, 4, 8

Infinite: 1, 3, 5, 7, ..., 2n-1, ...

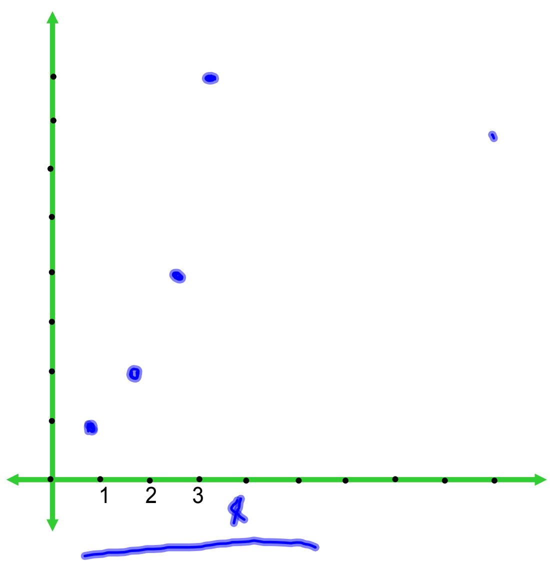
$a_1, a_2, a_3, \dots$

$a_n = 2n - 1$

$n = 1 \rightarrow a_1 = 2(1) - 1 = 1$

$n = 2 \rightarrow a_2 = 2(2) - 1 = 3$

A sequence is a function with the domain a subset of the natural numbers.



1, 2, 4, 8

Example 1:

a) Write the first four terms of this sequence: $a_n = n^2 + 1$

$a_1 = 1^2 + 1 = 2$

$a_2 = 2^2 + 1 = 5$

$a_3 = 3^2 + 1 = 10$

$a_4 = 17$

2, 5, 10, 17

b) Write the first four terms of this sequence: $b_n = (-1)^{n+1}(10n + 3)$

$b_1 = (-1)^2(10+3) = 13$

$b_2 = (-1)^3(10\cdot 2+3) = -23$

$b_3 = 33$ $b_4 = -43$

13, -23, 33, -43

Example 2: Find a formula for the n^{th} term in each of these sequences, then use the formula to find the 10^{th} term.

a) 2, 4, ~~5~~ 6, 8, 10, ...

$a_n = 2n$

$a_{10} = 2(10) = 20$

b) 3, -6, 12, -24, ...

$3\cdot1, -3\cdot2, 3\cdot4, -3\cdot8$

$b_n = (-1)^{n+1}(3\cdot2^{n-1}) = (-1)^{n+1}(3\cdot2^{n-1})$

$b_{10} = (-1)^{11}(3\cdot2^{9})$

Some sequences are defined ***recursively***. One or more initial terms are given and subsequent terms are defined using the previous terms.

Example 3:

$a_1 = 2$ $a_n = 3a_{n-1} + 1$ for each $n > 1$

What are the first four terms?

$a_1, a_2, a_3, a_4, \ldots$

$2, 7, 22, 67$

$a_1 = 2$

$a_2 = 3(2) + 1 = 7$

$a_3 = 3(7) + 1 = 22$

$a_4 = 3(22) + 1 = 67$

Example 4:

The ***Fibonacci Sequence***

$a_1 = 1$ ✓

$a_2 = 1$ ✓

$a_k = a_{k-1} + a_{k-2}$ ✓

$a_3 = a_1 + a_2 = 1+1 = 2$

List five terms:

a_1, a_2, a_3, a_4, a_5

1, 1, 2, 3, 5, 8

Factorials are often used in sequence definitions.

We define *n factorial* (written *n!*) to be:

$n! = 1\cdot 2\cdot 3\cdot \ldots \cdot (n-1)\cdot n$

$4! = 1\cdot 2\cdot 3\cdot 4$

$= 24$

0! is defined to be 0! = 1

Example 5:

Evaluate these expressions:

a) $\frac{8!}{2!\cdot 6!} = \frac{\cancel{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\cdot 7\cdot 8}{(1\cdot 2)(\cancel{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6})} =$

$= \frac{56}{2} = 28$

b) $\frac{(n+1)!}{(n-1)!} = \frac{\cancel{1\cdot 2\cdot 3\cdots(n-1)}(n)(n+1)}{\cancel{1\cdot 2\cdots(n-1)}}$

$= \frac{n(n+1)}{1} = n^2+n$

It is often convenient to recognize the factorials of the first five or six natural numbers.

$1! = 1$

$2! = 1\cdot 2 = 2$

$3! = 6$

$4! = 24$

$5! = 120$

$6! = 720$

Example 6:

Write the first four terms of these sequences:

a) $a_n = \frac{1}{n!}$

$a_1 = 1/1! = 1$

$a_2 = 1/2! = 1/2$

$a_3 = 1/3! = 1/6$

$a_4 = 1/4! = 1/24$

$1, 1/2, 1/6, 1/24$

b) $b_n = \frac{n}{(n+2)!}$

$b_1 = 1/3! = 1/6$

$b_2 = 2/4! = 2/24$

$b_3 = 3/5! = 3/120$

$b_4 = 4/6! = 4/720$

$1/6, 1/12, 1/40, 1/180$

A ***series*** is the sum of the terms in a sequence. The sum of the first *n* terms of a sequence is the n^{th} *partial sum* S_n.

The 5^{th} partial sum of the sequence of odd numbers is $S_5 = 1+3+5+7+9$

$= 25$

For an arbitrary sequence $a_1, a_2, a_3, \ldots, a_{100}$, the corresponding series is

$a_1 + a_2 + a_3 + \ldots + a_{100}$.

We abbreviate this sum using the Greek letter Σ (sigma):

$$\sum_{i=1}^{100} a_i = a_1 + a_2 + a_3 + \ldots + a_{100}.$$

The subscript i=1 and superscript 100 written above and below sign indicate which terms begin and end the series. The index i is not unique, but is sometimes replaced using j, k, etc.

Express $3^2 + 4^2 + 5^2 + 6^2$ using the sigma.

$$S_4 = \sum_{j=3}^{6} j^2$$

Example 7:

Find the sum of these series by adding the terms:

a) $\sum_{j=1}^{5} (1+3j)$

$a_j = 1 + 3j$

$j = 1 \rightarrow j = 5$

$4 + 7 + 10 + 13 + 16 = 50$

b) $\sum_{k=0}^{2} \frac{(-1)^k}{2^k}$

$b_k = \frac{(-1)^k}{2^k}$

$k = 0 \rightarrow k =$

$\frac{1}{1} + \frac{-1}{2} + \frac{1}{4} =$

$1 - \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

Use summation notation to abbreviate this series:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \; + \ldots + \frac{1}{99\cdot 100}$$

$$\sum_{i=1}^{99} \frac{1}{i(i+1)}$$

Example 8:

You're a clever student. You've decided to save your money for a trip to Europe, but it will be expensive. You've decided to open a savings account today with $1. You plan to add more each day, 7 days a week, by depositing one more dollar each day than you did the previous day. Use summation notation to express the total amount you will have contributed at the end of one year:

$$1+2+3+4+\cdots+365=$$

$$\sum_{j=1}^{365} j$$