

## 9.5 The Binomial Theorem

- \* Use the Binomial Theorem to calculate binomial coefficients.
- \* Use Pascal's Triangle to calculate binomial coefficients.
- \* Find the  $n$ th term in a binomial expansion.

$$\begin{array}{l} (3x - 2y)^6 \\ (2x + 5)^8 \end{array} \rightarrow 6^{\text{th}} \text{ term} \quad \begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \end{array}$$

What does the word binomial mean?

$$3x-2 \quad x^2+y \quad x-2y$$

$$(a+b)^0 = 1$$

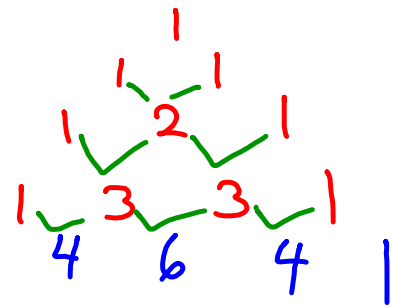
$$(a+b)^1 = a + b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

Pascal's triangle

coefficients



$$(a+b)(a^2+2ab+b^2) = a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3$$

$$(a+b)^4$$

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$$a^3 + 3a^2b + 3ab^2 + b^3$$

What does 7! mean? 7 Factorial  $\rightarrow 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Example 1: Determine the value of each of these.

a)  $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$

b)  $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3,628,800$

c)  $12!/10! = \frac{12 \cdot 11 \cdot \cancel{10!}}{\cancel{10!}} = 132$

d)  $n!$

e)  $(n+2)! = n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

f)  $0! = (n+2)(n+1) \cdots 3 \cdot 2 \cdot 1$

$= 1$  By def.

Example 2: A pizza shop offers 4 different toppings, Onions, Mushrooms, Pepperoni and Ham. How many 'different' pizzas can you order having none, one, two, three or all four toppings?

$\{ \}$  1  
 $\{O\} \{M\} \{P\} \{H\}$  4  
 $\{O, M\} \{O, P\} \{O, H\} \{M, P\} \{M, H\} \{P, H\}$  6  
 $\{M, P, H\} \{O, P, H\} \{O, M, H\} \{O, M, P\}$  4  
 $\{O, M, P, H\}$  1

Combination of  $n$  things taken  $r$  at a time.

What does  ${}^n C_r$  mean? I have  $n$  things, I choose  $r$  of them.

$${}^n C_r = \frac{n!}{(n-r)!r!} = \binom{n}{r} = {}^n C_r$$

Determine the value of each of these.

${}^4 C_0$	${}^4 C_1$	${}^4 C_2$	${}^4 C_3$	${}^4 C_4$
$\frac{4!}{4! \cdot 0!}$	$\frac{4!}{3! \cdot 1!}$	$\frac{4!}{2! \cdot 2!}$	$\frac{4!}{1! \cdot 3!}$	$\frac{4!}{0! \cdot 4!}$
1	$\frac{4 \cdot 3!}{3!}$	$\frac{4 \cdot 3 \cdot 2!}{2! \cdot 2}$	4	1
	4	6		

Example 3: Determine the value of each of these and make up a question it might answer.

$${}^6C_2 \quad \frac{6!}{4!2!} = \frac{6 \cdot 5 \cdot \cancel{4!}}{\cancel{4!} \cdot 2} = 15$$

How many ways can I select:  
2 friends out of 6 to take to dinner.

$${}^{12}C_{10} \quad \frac{12!}{2!10!} = \frac{12 \cdot 11 \cdot \cancel{10!}}{2 \cdot \cancel{10!}} = 66$$

10 of 12 books to read this summer?

$${}^7C_4 \quad \frac{7!}{3!4!} = \frac{7 \cdot 6 \cdot 5 \cdot \cancel{4!}}{3 \cdot 2 \cdot \cancel{4!}} = 35$$

4 of 7 coins to give away.

$${}^{15}C_0 \quad \frac{15!}{15!0!} = 1$$

0 out of 15 cards to give away.

**Binomial Theorem** and Pascal's Triangle

$$(x+y)^n = \sum_{j=0}^n \binom{n}{j} x^{n-j} y^j$$

0<sup>th</sup> row

Pattern

5<sup>th</sup> row

Mathematics

$\binom{0}{0}$   
 $\binom{1}{0} \binom{1}{1}$   
 $\binom{2}{0} \binom{2}{1} \binom{2}{2}$   
 $\binom{3}{0} \binom{3}{1} \binom{3}{2} \binom{3}{3}$   
 $\binom{4}{0} \binom{4}{1} \binom{4}{2} \binom{4}{3} \binom{4}{4}$   
 $\binom{5}{0} \binom{5}{1} \binom{5}{2} \binom{5}{3} \binom{5}{4} \binom{5}{5}$

$\frac{5!}{2!3!} = \frac{5 \cdot 4 \cdot 3!}{2! \cdot 3!} = 10$

So,  $(a+b)^5 = 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$

$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

Example 4: Expand this binomial.  $(2x - y)^4 =$

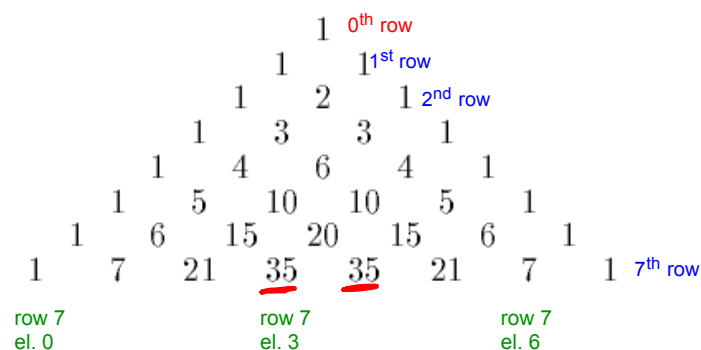
$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4 \quad \checkmark$$

$$a = 2x \quad b = -y \quad \checkmark$$

$$(2x)^4 + 4(2x)^3(-y) + 6(2x)^2(-y)^2 + 4(2x)(-y)^3 + (-y)^4$$
$$16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$$

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Pascal's Triangle



Sum of n<sup>th</sup> row

row	Sum
0	1 = 2 <sup>0</sup>
1	2 = 2 <sup>1</sup>
2	4 = 2 <sup>2</sup>
3	8 = 2 <sup>3</sup>
4	16 = 2 <sup>4</sup>
	⋮
n <sup>th</sup>	2 <sup>n</sup>

$\binom{7}{0} \binom{7}{1} \binom{7}{2} \binom{7}{3} \dots \binom{7}{7}$

Example 5: How do we find the  $x^6$  term in the expansion of  $(2x-y)^{10}$  without writing the entire expansion?

$$\binom{10}{4} (2x)^6 (-y)^4 = \frac{10!}{6!4!} (2x)^6 (-y)^4$$

$$210 \cdot 64x^6y^4 = 13,440x^6y^4$$



Example 6: An interesting application of Pascal's Triangle is in probability.  
 In a family of six children, what is the probability that two are boys and the rest are girls?

$$\begin{array}{cccccccc}
 1 & 6 & 15 & 20 & 15 & 6 & 1 & \text{Sum } 2^6 \\
 & & & & & & & \underline{64} \\
 & \swarrow & \rightarrow & \rightarrow & & & & \\
 & \text{no boys} & \text{1 boy} & \text{2 boys} & & & & \\
 \binom{6}{0} & \binom{6}{1} & \binom{6}{2} & \dots & & & & \\
 & & \text{circled} & & & & & 
 \end{array}$$

$$P(\text{2 boys in 6 children}) = \frac{\binom{6}{2}}{2^6} = \frac{15}{64} \approx 0.234 \approx 23\%$$