9.5 The Binomial Theorem

* Use the Binomial Theorem to calculate binomial coefficients.

* Use Pascal's Triangle to calculate binomial coefficients.

* Find the nth term in a binomial expansion.

$(3x-2y)^6$

$(2x+5)^8 \rightarrow 6^{th}$ term

1

1 1

1 2 1

What does the word <u>binomial</u> mean?

$3x-2 \qquad x^2+y \qquad x-2y$

Pascal's triangle

Coefficients

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

$(a+b)^0 = 1$

$(a+b)^1 = a+b$

$(a+b)^2 = a^2+2ab+b^2$

$(a+b)^3 = a^3+3a^2b+3ab^2+b^3$

$$
\begin{aligned}
(a+b)(a^2+2ab+b^2) = a^3 &+ 2a^2b + ab^2 \\
& a^2b + 2ab^2 + b^3 \\
\hline
a^3 &+ 3a^2b + 3ab^2 + b^3
\end{aligned}
$$

✓

$(a+b)^4$

What does 7! mean? 7 Factorial $\rightarrow 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

Example 1: Determine the value of each of these.

a) 4! $= 4 \cdot 3 \cdot 2 \cdot 1 = 24$

b) 10! $= 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 3{,}628{,}800$

c) 12!/10! $= \dfrac{12 \cdot 11 \cdot \cancel{10!}}{\cancel{10!}_{1}} = 132$

d) n ! $= n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1$

e) (n+2)! $= (n+2)(n+1) \cdots 3 \cdot 2 \cdot 1$

f) 0! $= 1$ By def.

Example 2: A pizza shop offers 4 different toppings, Onions, Mushrooms, Pepperoni and Ham. How many 'different' pizzas can you order having none, one, two, three or all four toppings?

{ } 1

{O} {M} {P} {H} 4

{O, M} {O, P} {O, H} {M, P} {M, H} {P, H} 6

{M, P, H} {O, P, H} {O, M, H} {O, M, P} 4

{O, M, P, H} 1

Combination of n things taken r at a time.

What does $_nC_r$ mean? I have n things, I choose r of them.

$$_nC_r = \frac{n!}{(n-r)!r!} = \binom{n}{r} = {}^nC_r$$

Determine the value of each of these.

$_4C_0$	$_4C_1$	$_4C_2$	$_4C_3$	$_4C_4$
$\frac{4!}{4!\cdot 0!}$	$\frac{4!}{3!1!}$	$\frac{4!}{2!2!}$	$\frac{4!}{1!3!}$	$\frac{4!}{0!\cdot 4!}$
1	$\frac{4\cdot \cancel{3!}}{\cancel{3!}}$	$\frac{4\cdot 3\cdot \cancel{2!}}{\cancel{2!}\cdot 2}$	4	1
	4	6		

Example 3: Determine the value of each of these and make up a question it might answer.

How many ways can I select:

${}_6C_2$ $\quad \dfrac{6!}{4!\,2!} = \dfrac{6\cdot 5\cdot \cancel{4!}}{\cancel{4!}\cdot 2} = 15$

2 friends out of 6 to take to dinner.

${}_{12}C_{10}$ $\quad \dfrac{12!}{2!\,10!} = \dfrac{12\cdot 11\cdot \cancel{10!}}{2\cdot \cancel{10!}} = 66$

10 of 12 books to read this summer?

${}_7C_4$ $\quad \dfrac{7!}{3!\,4!} = \dfrac{7\cdot \cancel{6}\cdot 5\cdot \cancel{4!}}{\cancel{3}\cdot \cancel{2}\cdot \cancel{4!}} = 35$

4 of 7 coins to give away.

${}_{15}C_0$ $\quad \dfrac{\cancel{15}!}{\cancel{15}!\cdot 0!} = 1$

0 out of 15 cards to give away.

Binomial Theorem and Pascal's Triangle

$$(x+y)^n = \sum_{j=0}^{n} \binom{n}{j} x^{(n-j)} y^{j}$$

Pattern

0th row

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

5th row

Mathematics

$$
\begin{array}{ccccccccccc}
 & & & & & \binom{0}{0} & & & & & \\
 & & & & \binom{1}{0} & & \binom{1}{1} & & & & \\
 & & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & & \\
 & & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & & \\
 & \binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4} & \\
\binom{5}{0} & & \binom{5}{1} & & \binom{5}{2} & & \binom{5}{3} & & \binom{5}{4} & & \binom{5}{5}
\end{array}
$$

$$\frac{5!}{2!\,3!} = \frac{5 \cdot 4 \cdot \cancel{3!}}{2 \cdot \cancel{3!}} = 10$$

So, $(a+b)^5 = 1a^5b^0 + 5a^4b^1 + 10a^3b^2 + 10a^2b^3 + 5a^1b^4 + 1a^0b^5$

$$a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Example 4: Expand this binomial. $(2x - y)^4 =$

$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$ ✓

$a = 2x \quad b = -y$ ✓

$(2x)^4 + 4(2x)^3(-y) + 6(2x)^2(-y)^2 + 4(2x)(-y)^3 + (-y)^4$

$16x^4 - 32x^3y + 24x^2y^2 - 8xy^3 + y^4$

Pascal's Triangle

1 0^{th} row

1 1 1^{st} row

1 2 1 2^{nd} row

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1 7^{th} row

row 7 el. 0 — row 7 el. 3 — row 7 el. 6

$\binom{7}{0}\binom{7}{1}\binom{7}{2}\binom{7}{3}\ \cdots\ \binom{7}{7}$

Sum of nth row

row	Sum	
0	1	$= 2^0$
1	2	$= 2^1$
2	4	$= 2^2$
3	8	$= 2^3$
4	16	$= 2^4$
		$\vdots$
	n^{th}	2^n

Example 5: How do we find the x^{6th} term in the expansion of $(2x-y)^{10}$ without writing the entire expansion?

$$\binom{10}{4}(2x)^6(-y)^4 = \frac{10!}{6!\,4!}(2x)^6(-y)^4$$

$$210\cdot 64x^6y^4 = 13,440x^6y^4$$

Example 6: An interesting application of Pascal's Triangle is in probability.
In a family of six children, what is the probability that two are boys and the rest are girls?

1 6 15 20 15 6 1 Sum 2^6

$\underline{64}$

no boys $\binom{6}{0}$ 1 boy $\binom{6}{1}$ 2 boy $\binom{6}{2}$ ...

$$P(\text{2 boys in 6 children}) = \frac{\binom{6}{2}}{2^6} = \frac{15}{64} \approx 0.234 \approx 23\%$$