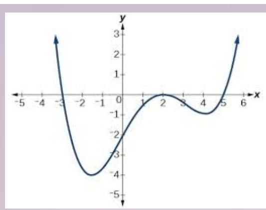
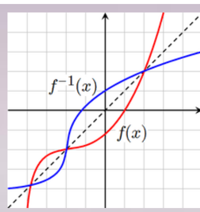

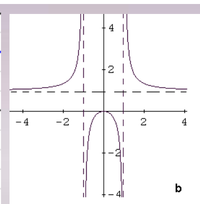

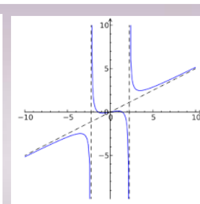
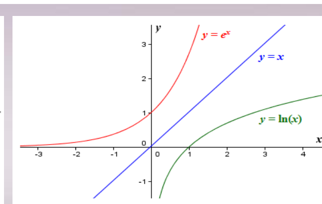

Math 1050 ~ College Algebra

25 Systems of Linear Equations: Matrix Inverses

$$-3x + 4y = 5$$
$$2x - y = -10$$
$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$
$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

Learning Objectives

- Find the inverse of a 2×2 or a 3×3 matrix.
- Solve a system of linear equations using an inverse matrix.

Inverse Matrix

If A and B are square matrices, $n \times n$, such that $AB = BA = I_n$, then B is the inverse matrix of A and can be denoted as A^{-1}. read as "A inverse" (-1 is NOT an exponent) (A^{-1} is multiplicative inverse of A)

Ex 1: Show that B is A^{-1}. $A = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$$AB = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 6-5 & -2+2 \\ 15-15 & -5+6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Process for finding an inverse matrix. (for a square matrix)

1. Augment A with I.
2. Perform row operations until the left side looks like I.
3. The right side will be A^{-1}.

Ex 2: Determine the inverse of each matrix, if it exists.

a) $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = A$

$$\left[\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right]$$

(-2): $-2 \quad -8 \quad 0 \quad -2$

$$\left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 2 & 3 & 1 & 0 \end{array}\right]$$

$$(-\tfrac{1}{5})\left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & -5 & 1 & -2 \end{array}\right]$$

$$(-4)\left[\begin{array}{cc|cc} 1 & 4 & 0 & 1 \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

$0 \quad -4 \quad \frac{4}{5} \quad -\frac{8}{5}$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{4}{5} & -\frac{3}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{2}{5} \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

b) $\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & -2 & -3 \end{bmatrix} = B$

$-1 \quad 1 \quad 0 \quad -1 \quad 0 \quad 0$

$$(-1)\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$-6 \quad 6 \quad 0 \quad -6 \quad 0 \quad 0$

$$(-6)\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 6 & -2 & -3 & 0 & 0 & 1 \end{array}\right]$$

$0 \quad -4 \quad 4 \quad 4 \quad -4 \quad 0$

$$(-4)\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 4 & -3 & -6 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & 1 \\ 0 & 1 & 0 & -3 & -3 & 1 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

$$B^{-1} = \begin{bmatrix} -2 & -3 & 1 \\ -3 & -3 & 1 \\ -2 & -4 & 1 \end{bmatrix}$$

Let's derive a formula for the inverse of a 2×2 matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

(assume $a \neq 0$)
$ad-bc \neq 0$

$$\frac{-bc}{a} + d = \frac{-bc}{a} + \frac{ad}{a} = \frac{ad-bc}{a}$$

$$\begin{array}{c} \\ \left(\frac{-c}{a}\right) \\ \\ \end{array}
\begin{array}{cccc} -c & \frac{-bc}{a} & \frac{-c}{a} & 0 \end{array}$$

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \left(\frac{a}{ad-bc}\right) \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & \frac{ad-bc}{a} & \frac{-c}{a} & 1 \end{array}\right]$$

$$(-b)\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\begin{array}{cccc} 0 & -b & \frac{bc}{ad-bc} & \frac{-ab}{ad-bc} \end{array}$$

$$\left(\frac{1}{a}\right)\left[\begin{array}{cc|cc} a & 0 & \frac{ad}{ad-bc} & \frac{-ab}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

$$\left[\begin{array}{cc|cc} 1 & 0 & \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ 0 & 1 & \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right]$$

aside

$$\frac{bc}{ad-bc} + 1 = \frac{\cancel{bc}}{ad-bc} + \frac{ad-\cancel{bc}}{ad-bc} = \frac{ad}{ad-bc}$$

$$\boxed{A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}$$

formula for A^{-1}, given $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ if $ad-bc \neq 0$.

We can write a system of linear equations as a matrix equation

$\boxed{AX = C,}$ where A is a matrix of coefficients, X is the matrix of variables and C is the matrix of constants.

Ex 3: Write this system of equations as a matrix equation.

$2x + y = 4$

$5x + 3y = 6$

$$A=\begin{bmatrix}2 & 1\\5 & 3\end{bmatrix} \quad X=\begin{bmatrix}x\\y\end{bmatrix} \quad C=\begin{bmatrix}4\\6\end{bmatrix}$$

$$AX=\begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}2x+y\\5x+3y\end{bmatrix}=\begin{bmatrix}4\\6\end{bmatrix}$$

Ex 4: Using this fact from Ex. 1, $A=\begin{bmatrix}2 & 1\\5 & 3\end{bmatrix}$ $A^{-1}=\begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix}$

find the solution to Ex. 3.

$$AX=B$$

$$\underbrace{A^{-1}A}X=A^{-1}B$$

$$IX=A^{-1}B$$

$$\boxed{X=A^{-1}B}$$

assuming A^{-1} exists.

$$X=A^{-1}B$$

$$X=\begin{bmatrix}3 & -1\\-5 & 2\end{bmatrix}\begin{bmatrix}4\\6\end{bmatrix}$$

$$=\begin{bmatrix}3(4)+-1(6)\\-5(4)+2(6)\end{bmatrix}=\begin{bmatrix}6\\-8\end{bmatrix}$$

$\boxed{x=6,\ y=-8}$ or pt $(6,-8)$

Ex 5: Refer back to example 2 to solve these systems of equations.

a) $2x+3y=0$
$x+4y=-5$

$A=\begin{bmatrix}2 & 3\\1 & 4\end{bmatrix}$ $A^{-1}=\frac{1}{5}\begin{bmatrix}4 & -3\\-1 & 2\end{bmatrix}$

$AX=B$ $B=\begin{bmatrix}0\\-5\end{bmatrix}$

$X=A^{-1}B=\frac{1}{5}\begin{bmatrix}4 & -3\\-1 & 2\end{bmatrix}\begin{bmatrix}0\\-5\end{bmatrix}$

$X=\frac{1}{5}\begin{bmatrix}0+15\\0+-10\end{bmatrix}=\frac{1}{5}\begin{bmatrix}15\\-10\end{bmatrix}=\begin{bmatrix}3\\-2\end{bmatrix}$

$\boxed{(3,-2)}$

b) $x-y=2$
$x-z=3$
$6x-2y-3z=15$

$A=\begin{bmatrix}1 & -1 & 0\\1 & 0 & -1\\6 & -2 & -3\end{bmatrix}$

$B=\begin{bmatrix}2\\3\\15\end{bmatrix}$ $A^{-1}=\begin{bmatrix}-2 & -3 & 1\\-3 & -3 & 1\\-2 & -4 & 1\end{bmatrix}$

$X=\begin{bmatrix}x\\y\\z\end{bmatrix}=A^{-1}B$

$=\begin{bmatrix}-2 & -3 & 1\\-3 & -3 & 1\\-2 & -4 & 1\end{bmatrix}\begin{bmatrix}2\\3\\15\end{bmatrix}=\begin{bmatrix}2\\0\\-1\end{bmatrix}$

$\boxed{(2,0,-1)}$

Ex 6: Solve this system using the techniques of this lesson.

$2x-3y=8$
$-4x+6y=-5$

$X=A^{-1}B$

① $2x-3y=8$
② $-4x+6y=-5$
$\Longleftrightarrow 2x-3y=\frac{5}{2}$
(parallel lines)

$A=\begin{bmatrix}2 & -3\\-4 & 6\end{bmatrix}$ $B=\begin{bmatrix}8\\-5\end{bmatrix}$

Find A^{-1}, if we can.

$A^{-1}=\frac{1}{2(6)-(-3)(-4)}\begin{bmatrix}6 & 3\\4 & 2\end{bmatrix}$

WARNING: we are trying to divide by 0!!!

$\Rightarrow A^{-1}$ DNE
$\Rightarrow$ for this system of eqns, there's $\boxed{\text{N.S.}}$