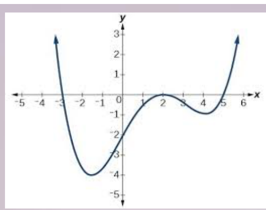
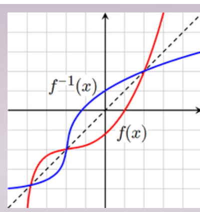
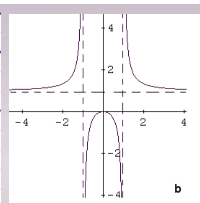
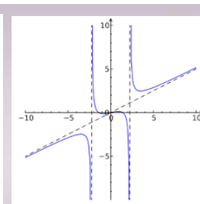
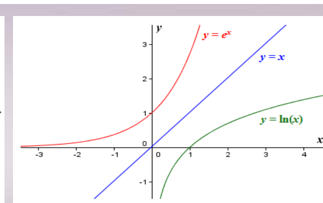










Math 1050 ~ College Algebra

5 Inverses of Functions

$$-3x + 4y = 5$$
$$2x - y = -10$$
$$\begin{bmatrix} -3 & 4 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$$

$$\sum_{k=1}^{m} k = \frac{m(m+1)}{2}$$
$$\sum_{k=0}^{n} z^k = \frac{1 - z^{n+1}}{1 - z}$$

Learning Objectives

- Verify that two functions are inverses of each other.
- Determine if a function is one-to-one.
- Use the graph of a one-to-one function to graph its inverse function.
- Find the inverse of a one-to-one function.

Inverse Function

If f and g are functions such that

- $(f \circ g)(x) = x$ for all x in the domain of g
- $(g \circ f)(x) = x$ for all x in the domain of f

then f and g are inverses of each other. (i.e. f and g "undo" each other)

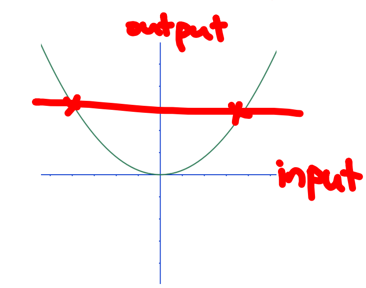
This is written $f^{-1}(x) = g(x)$ and $g^{-1}(x) = f(x)$.

read "f inverse of x"

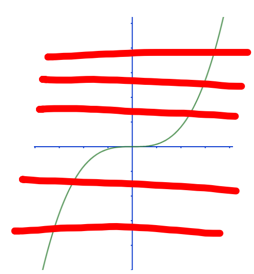
note: the -1 is NOT an exponent.

To have an inverse, a function must be one-to-one, that is for each output there must be exactly one input.

not one-to-one



is one-to-one



horizontal line test: if every horizontal line crosses a function's graph only once (or zero times), then the function is one-to-one, i.e. an inverse exists.

Finding an Inverse Function

Strategy given $y = f(x)$

① swap x and y. $x = f(y)$

② Do legal algebra in order to solve the equation for y in terms of x.

③ your answer will be $y = f^{-1}(x)$.

Ex 1: For $f(x)$, find the inverse function, $f^{-1}(x)$.

a) $f(x) = \frac{x^5 - 1}{3}$ $y = \frac{x^5 - 1}{3}$

① $x = \frac{y^5 - 1}{3}$

② $3x = y^5 - 1$

$(3x+1)^{1/5} = (y^5)^{1/5}$

$\sqrt[5]{3x+1} = y$

③ $f^{-1}(x) = \sqrt[5]{3x+1}$

b) $f(x) = \sqrt[3]{x+2} + 1$ $y = \sqrt[3]{x+2} + 1$

① $x = \sqrt[3]{y+2} + 1$

② $(x-1)^3 = \left(\sqrt[3]{y+2}\right)^3$

$(x-1)^3 = y + 2$

$(x-1)^3 - 2 = y$

③ $f^{-1}(x) = (x-1)^3 - 2$

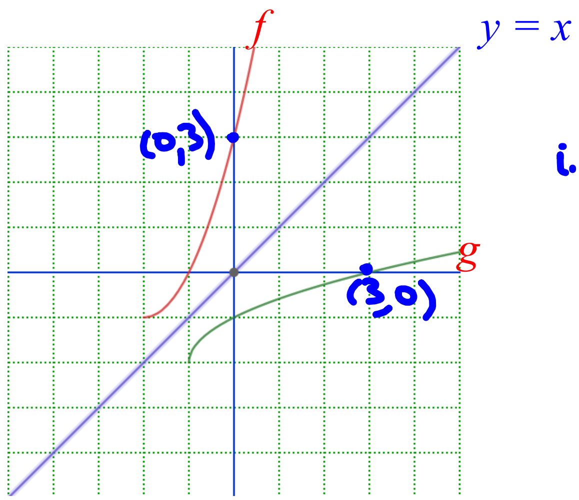
Graphical Properties of Inverse Functions

Assume f and g are inverse functions.

- The domain of f is the range of g and the domain of g is the range of f.
- $f(a) = b$ if and only if $g(b) = a$.
- (a,b) is on the graph of f if and only if (b,a) is on the graph of g.
- f and g are symmetric about the line $y = x$.

on $f(x)$: pt (a,b)

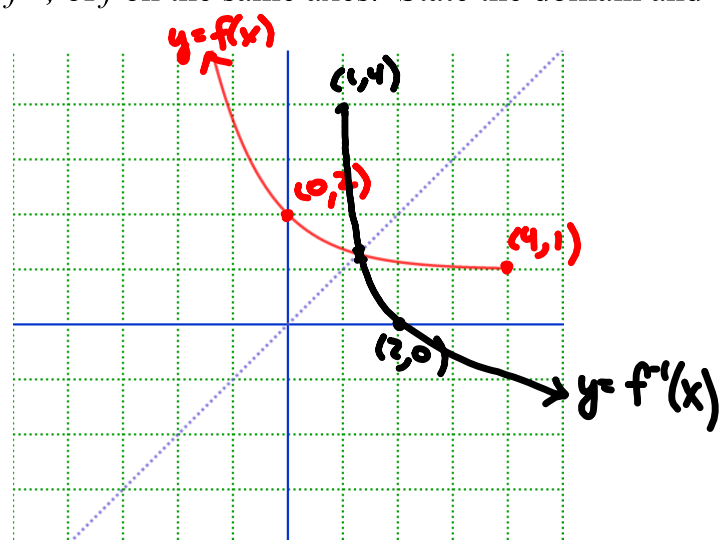
on $f^{-1}(x)$: pt (b,a)



i.e. the graph of $y=f(x)$ and $y=f^{-1}(x)$ are mirror images of each other across the line $y=x$.

Ex 2: Sketch the inverse, f^{-1}, of f on the same axes. State the domain and range of each.

	Domain	Range
$f(x)$	$(-\infty,4]$	$[1,\infty)$
$f^{-1}(x)$	$[1,\infty)$	$(-\infty,4]$



Ex 3: Show that these two functions are inverses in two ways.

$g(x) = \dfrac{1-x}{x}, \quad 0 < x \le 1 \qquad f(x) = \dfrac{1}{1+x}, \quad x \ge 0$

base: $y = \frac{1}{x}$ shift left 1 $(0,1)$

$g(x) = \frac{1}{x} - 1$ shift down 1

base: $y = \frac{1}{x}$ $0 < x \le 1$

$(1,0)$

a) Algebraically

Prove ① $g(f(x)) = x \quad (x \ge 0)$

and ② $f(g(x)) = x \quad (0 < x \le 1)$.

① $g(f(x)) = g\left(\frac{1}{1+x}\right)$

$= \left[\dfrac{1 - \frac{1}{1+x}}{\frac{1}{1+x}}\right]\left(\dfrac{1+x}{1+x}\right)$

$= \dfrac{1+x-1}{1} = 1+x-1 = x$ ✓

② $f(g(x)) = f\left(\frac{1-x}{x}\right)$

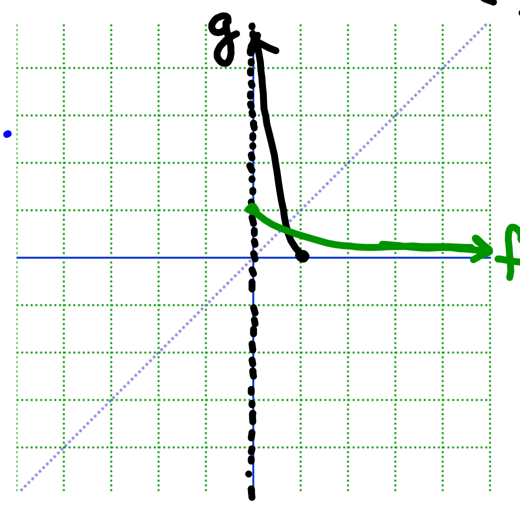
$= \dfrac{1}{1+\left(\frac{1-x}{x}\right)} = \dfrac{1}{\cancel{1} + \frac{1}{x} - \cancel{1}} = \dfrac{1}{1/x}$

$= x$ ✓

because $f(g(x)) = x = g(f(x))$ (for x in their domains)

f and g are inverse fns.

b) Graphically



Note: $\dfrac{1-x}{x}$

$= \dfrac{1}{x} - \dfrac{x}{x} = \dfrac{1}{x} - 1$

Ex 4: Find the inverse of $f(x) = \frac{x-3}{x+2}$. $y = \frac{x-3}{x+2}$

① $x = \frac{y-3}{y+2}$

② $(y+2)x = \left(\frac{y-3}{\cancel{y+2}}\right)(\cancel{y+2})$

$xy + 2x = y - 3$

$xy - y + 2x = -3$

$xy - y = -2x - 3$

$\frac{y\cancel{(x-1)}}{\cancel{(x-1)}} = \frac{-2x-3}{(x-1)}$

$y = \frac{-2x-3}{x-1}$

③ $f^{-1}(x) = \frac{-2x-3}{x-1}$

	D	R
$f(x)$	$(-\infty,-2) \cup (-2,\infty)$	$(-\infty,1) \cup (1,\infty)$
$f^{-1}(x)$	$(-\infty,1) \cup (1,\infty)$	$(-\infty,-2) \cup (-2,\infty)$