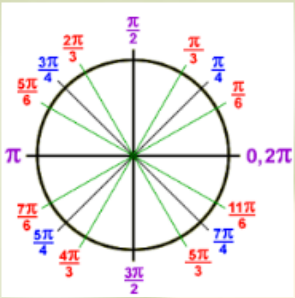
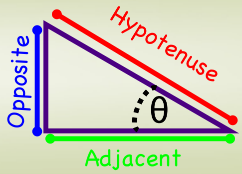
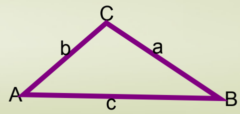
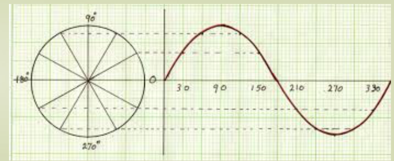








Math 1060 ~ Trigonometry

23 The Dot Product

$\boxed{\sin^2 u + \cos^2 u = 1}$

$\boxed{\sin 2u = 2 \sin u \cos u}$

$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}}$

$\boxed{c^2 = a^2 + b^2 - 2ab\cos C}$

Learning Objectives

In this section you will:

- **Find the dot product of two vectors.**
- **Learn properties of the dot product.**
- **Determine the angle between two vectors.**
- **Determine whether or not two vectors are orthogonal.**
- **Solve applications of the dot product.**

Dot Product

The dot product of two vectors is a scalar. It can be useful in finding the angle between two vectors.

If $\boldsymbol{v} = \langle v_1, v_2 \rangle$ and $\boldsymbol{w} = \langle w_1, w_2 \rangle$, then $\boxed{\boldsymbol{v} \bullet \boldsymbol{w} = v_1 w_1 + v_2 w_2}$

Note: $\vec{w} \cdot \vec{w} = w_1 w_1 + w_2 w_2 = w_1^2 + w_2^2 = \|\vec{w}\|^2$ i.e. $\boxed{\|\vec{w}\|^2 = \vec{w} \cdot \vec{w}}$

Ex 1: Find the dot product of these pairs of vectors.

a) $\boldsymbol{v} = \langle 3,4 \rangle$ and $\boldsymbol{w} = \langle -2,5 \rangle$.

$\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v} = 3(-2) + 4(5)$

$= -6 + 20 = 14$

b) $\boldsymbol{v} = \langle -3,2 \rangle$ and $\boldsymbol{w} = \langle -4,-6 \rangle$

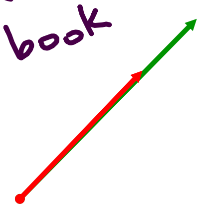
$\vec{v} \cdot \vec{w} = -3(-4) + 2(-6)$

$= 12 - 12 = 0$

Geometric Interpretation of the Dot Product

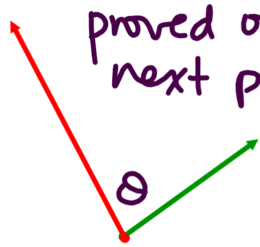
same direction

proved in book

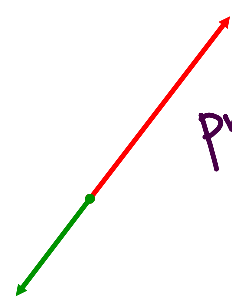


angle (non-reflexive)

proved on next page!



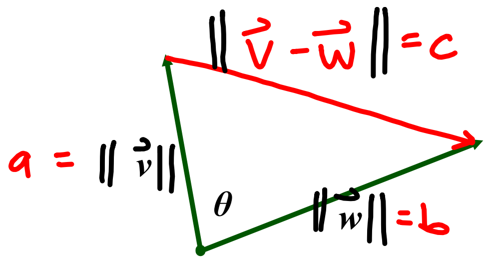
opposite directions



proved in book

Note: there are more dot product properties listed in the book to refer to.

We will use the Law of cosines to prove that claim $\boxed{v \bullet w = \|v\|\,\|w\|\cos\theta,\ 0 < \theta < \pi.}$



Pf Use law of Cosines. $c^2 = a^2 + b^2 - 2ab\cos\theta$

① $\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\,\|\vec{w}\|\cos\theta$

② we also know that $\|\vec{p}\|^2 = \vec{p}\cdot\vec{p}$.

$\Rightarrow \|\vec{v} - \vec{w}\|^2 = (\vec{v} - \vec{w})\cdot(\vec{v} - \vec{w})$

$= \vec{v}\cdot\vec{v} - \vec{v}\cdot\vec{w} - \vec{w}\cdot\vec{v} + \vec{w}\cdot\vec{w}$

$\vec{v}\cdot\vec{w} = \vec{w}\cdot\vec{v}$

$\|\vec{v} - \vec{w}\|^2 = \|\vec{v}\|^2 - 2\vec{w}\cdot\vec{v} + \|\vec{w}\|^2$

③ equate results from ① and ②

$\cancel{\|\vec{v}\|^2} + \cancel{\|\vec{w}\|^2} - 2\|\vec{v}\|\,\|\vec{w}\|\cos\theta = \cancel{\|\vec{v}\|^2} - 2\vec{w}\cdot\vec{v} + \cancel{\|\vec{w}\|^2}$

$\cancel{-2}\|\vec{v}\|\,\|\vec{w}\|\cos\theta = \cancel{-2}\vec{w}\cdot\vec{v}$

$\boxed{\vec{w}\cdot\vec{v} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta}$ $\theta \in (0, \pi)$

$\boldsymbol{v} \bullet \boldsymbol{w} = \|\boldsymbol{v}\| \, \|\boldsymbol{w}\| \cos \theta$ $\iff$ $\cos\theta = \dfrac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$

Ex 2: Determine the angle between these pairs of vectors.

a) $\boldsymbol{v} = \langle 3,4 \rangle$ and $\boldsymbol{w} = \langle -2,5 \rangle$.

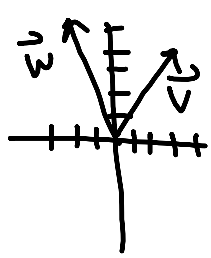
$\vec{v}\cdot\vec{w} = 3(-2)+4(5)$
$= -6+20 = 14$

$\|\vec{v}\| = \sqrt{3^2+4^2} = 5$

$\|\vec{w}\| = \sqrt{(-2)^2+5^2} = \sqrt{29}$

$\cos\theta = \dfrac{14}{5\sqrt{29}}$

$\theta = \cos^{-1}\left(\dfrac{14}{5\sqrt{29}}\right) \simeq 58.7^\circ$



b) $\boldsymbol{v} = \langle -3,2 \rangle$ and $\boldsymbol{w} = \langle -4,-6 \rangle$

$\vec{v}\cdot\vec{w} = -3(-4)+2(-6) = 0$

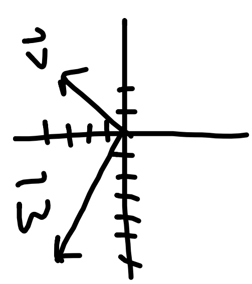
$\|\vec{v}\| = \sqrt{(-3)^2+2^2} = \sqrt{13}$

$\|\vec{w}\| = \sqrt{(-4)^2+(-6)^2} = \sqrt{52} = 2\sqrt{13}$

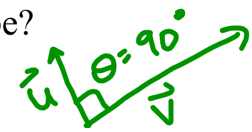
$\cos\theta = \dfrac{0}{\sqrt{13}\,(2\sqrt{13})} = 0$

$\theta = \cos^{-1}(0)$

$\theta = \dfrac{\pi}{2}$ or 90°



Orthogonal vectors: If two vectors are perpendicular to each other they are said to be orthogonal. What would the cosine of the angle between two orthogonal vectors be?



$\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}$

$\boxed{\text{if } \theta = 90°,\ \vec{u}\cdot\vec{v} = 0}$

Ex 3: Determine whether these pairs are vectors are orthogonal or not.

a) $\underset{}{\overset{\vec{u}}{\langle 3,-2\rangle}}$ and $\overset{\vec{v}}{\langle 1,4\rangle}$

$\vec{u}\cdot\vec{v} = 3(1) + -2(4)$

$= 3 - 8 = -5 \neq 0$

$\Rightarrow \vec{u}$ and $\vec{v}$ are NOT orthog.

b) $\overset{\vec{a}}{\langle 4,-6\rangle}$ and $\overset{\vec{b}}{\langle -3,-2\rangle}$

$\vec{a}\cdot\vec{b} = 4(-3) + -6(-2)$

$= -12 + 12 = 0$

$\Rightarrow \vec{a}$ and $\vec{b}$ ARE orthogonal.

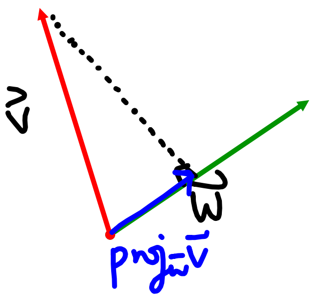
c) $\overset{\vec{p}}{\langle 2,-1\rangle}$ and $\overset{\vec{q}}{\langle -4, 2\rangle}$

$\vec{p}\cdot\vec{q} = 2(-4) + -1(2)$

$= -8 - 2 = -10 \neq 0 \Rightarrow \vec{p}$ and $\vec{q}$ are NOT orthog.

Orthogonal Projection

If $\boldsymbol{v}$ and $\boldsymbol{w}$ are nonzero vectors, then the orthogonal projection of $\boldsymbol{v}$ onto $\boldsymbol{w}$, denoted by $\text{proj}_w(\boldsymbol{v})$ is given by



$$proj_w(\vec v) = \left(\frac{\vec v \cdot \vec w}{\|\vec w\|^2}\right)\vec w = \left[\vec v \cdot \left(\frac{\vec w}{\|\vec w\|}\right)\right]\left(\frac{\vec w}{\|\vec w\|}\right)$$

$$proj_{\vec w}\vec v = \underbrace{(\vec v \cdot \hat w)}_{\text{scalar}}\hat w \quad \text{(this is a vector)}$$

vector in direction of $\vec w$

Ex 4: For $\boldsymbol{v} = \langle -6,-5\rangle$ and $\boldsymbol{w} = \langle 10,-8\rangle$, find $\text{proj}_w(\boldsymbol{v})$.

$$proj_{\vec w}\vec v = \left(\frac{\langle -6,-5\rangle \cdot \langle 10,-8\rangle}{10^2 + (-8)^2}\right)\langle 10,-8\rangle$$

$$= \frac{-60+40}{100+64}\langle 10,-8\rangle = \frac{-20}{164}\langle 10,-8\rangle$$

$$= \frac{-5}{41}\langle 10,-8\rangle = \left\langle \frac{-50}{41}, \frac{40}{41}\right\rangle$$

In physics, you will discover how this concept relates to problems about work.