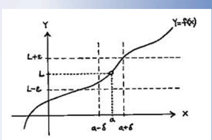
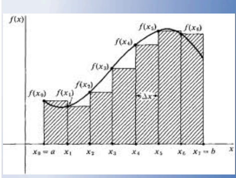


$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

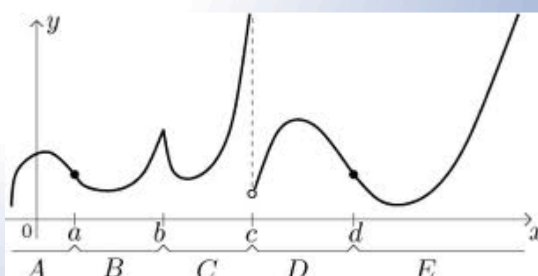
$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$



$$\lim_{max\,\Delta x_i \to 0} \sum_1^n f(x_i)\Delta x_i = \int_a^b f(x)dx$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Monotonicity and Concavity



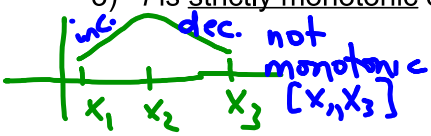
Definition

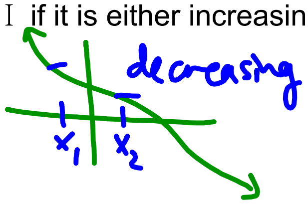
Let f be defined on an interval I, (open, closed or neither), we say that:

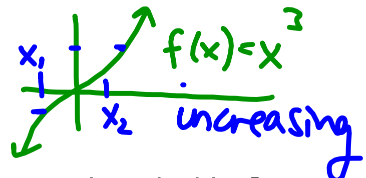
1) f is *increasing* on I if for every x_1, x_2 in I $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

2) f is *decreasing* on I if for every x_1, x_2 in I $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

3) f is strictly monotonic on I if it is either increasing or decreasing on I .







Monotonicity Theorem

Let f be continuous on the interval, I and differentiable everywhere inside I . (no singular pts on I)

1) if $f'(x) > 0$ for all x on the interval, then f is increasing on that interval.

2) if $f'(x) < 0$ for all x on the interval, then f is decreasing on that interval.

slope positive $\Leftrightarrow$ increasing
slope negative $\Leftrightarrow$ decreasing

EX 1 For each function, determine where *f* is increasing and decreasing.

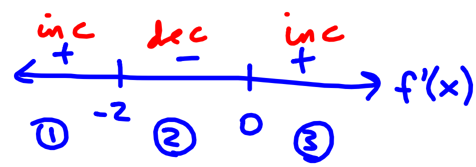
a) $f(x)=x^3+3x^2-12$ (cont everywhere)

$f'(x)=3x^2+6x=0$

$\boxed{3x(x+2)}=0$

$x=0,\ x=-2$

sign line



test values: $f'(x)=3x(x+2)$

① $x=-3$, $-(-)$

② $x=-1$, $-(+)$

③ $x=1$, $+(+)$

f increasing on $(-\infty,-2)\cup(0,\infty)$

f decreasing on $(-2,0)$

b) $f(x)=\dfrac{x-1}{x^2}$

(discontinuity at $x=0$) VA: $x=0$

$$f'(x)=\frac{x^2(1)-(x-1)(2x)}{x^4}$$

$$=\frac{x^2-2x^2+2x}{x^4}$$

$$=\frac{-x^2+2x}{x^4}=\frac{x(-x+2)}{x^4}=0$$

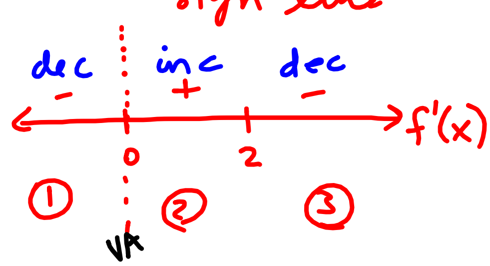
derivative DNE at $x=0$ (not a singular pt)

$$\frac{-x+2}{x^3}=0$$

$-x+2=0$

$x=2$ (stationary pt x-value)

sign line



$f'(x)=\dfrac{x(-x+2)}{x^4}$

① $x=-1$, $\dfrac{-(+)}{+}$

② $x=1$, $\dfrac{+(+)}{+}$

③ $x=3$, $\dfrac{+(-)}{+}$

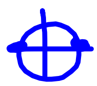
f is increasing $(0,2)$

f is decreasing $(-\infty,0)\cup(2,\infty)$

EX 2 Where is $f(x) = cos^2 x$ increasing and decreasing on the interval [**0**,**2**π]?

$f'(x) = 2\cos x(-\sin x) = -2\cos x \sin x$ (no singular pts)

$-2\cos x \sin x = 0$



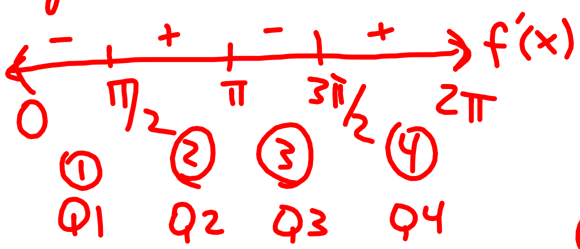
$-\sin(2x) = 0$

$\sin(2x) = 0$

$2x = 0, \pi, 2\pi, 3\pi, 4\pi$

$x = 0, \pi/2, \pi, 3\pi/2, 2\pi$ (x-values for stationary pts)

sign line



test: $f'(x) = -2\sin x \cos x$

① $x = \pi/4$, $-2(+)(+)$

② Q2, $-2(+)(-)$

③ Q3, $-2(-)(-)$

④ Q4, $-2(-)(+)$

f is increasing on $(\pi/2, \pi) \cup (3\pi/2, 2\pi)$

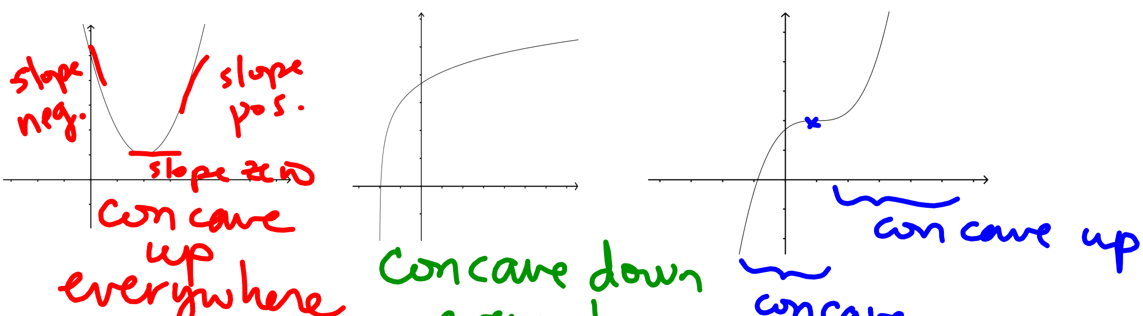
f is decreasing on $(0, \pi/2) \cup (\pi, 3\pi/2)$

Definition

Let f be differentiable on an open interval, I .

f is concave up on I if f'(x) is increasing on I , and

f is concave down on I if f'(x) is decreasing on I .



Concavity Theorem

Let f be twice differentiable on an open interval, I.

If $f''(x) > 0$ for all x on the interval, then f is concave up on the interval.

If $f''(x) < 0$ for all x on the interval, then f is concave down on the interval.

EX 3 Determine where this function is $\underbrace{\text{increasing, decreasing,}}_{f'(x)}$ $\underbrace{\text{concave up and concave down}}_{f''(x)}$.

$$f(x) = 4x^3 - 3x^2 - 6x + 12$$

$$f'(x) = 12x^2 - 6x - 6 = 0 \qquad \text{(no singular pts)}$$

$$6(2x^2 - x - 1) = 0$$

factored form of $f'(x)$: $\underline{6(2x+1)(x-1)} = 0$

$$2x+1=0 \text{ or } x-1=0 \Rightarrow x = -\tfrac{1}{2}, 1$$

sign line $f'(x)$: $+$ | $-$ | $+$ (marks at $-\frac{1}{2}$ and 1)

inc. $-\frac{1}{2}$ dec. 1 inc.

test values:

$x = -1$, $+(-)(-)$

$x = 0$, $+(+)(-)$

$x = 2$, $+(+)(+)$

$$f''(x) = \boxed{24x - 6} = 0$$

$$24x = 6$$

$$x = \tfrac{1}{4}$$

$f''(x)$: $-$ | $+$ (mark at $\frac{1}{4}$)

concave down $\frac{1}{4}$ concave up

test values

$x = 0$, $-$

$x = 1$, $24(1) - 6$

Answer:

increasing $(-\infty, -\frac{1}{2}) \cup (1, \infty)$

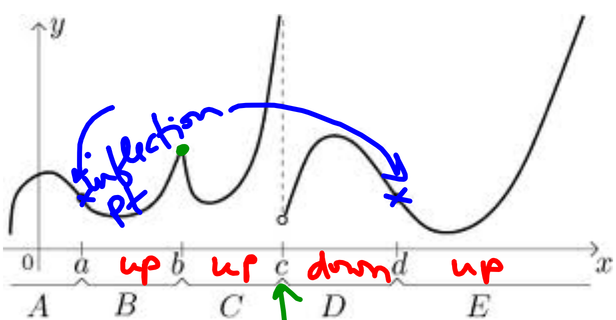
decreasing $(-\frac{1}{2}, 1)$

concave up $(\frac{1}{4}, \infty)$

concave down $(-\infty, \frac{1}{4})$

Inflection Point

Let f be continuous at c. We call $(c, f(c))$ an inflection point of f if f is concave up on one side of c and concave down on the other side of c.



green pt is not an inflectn pt (it's a singular pt)

concave down

VA (not an inflectn pt)

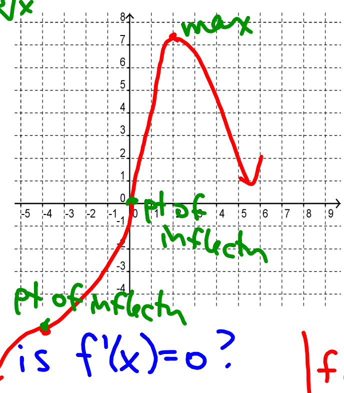
Inflection points will occur at x-values for which $f''(x) = 0$ or $f''(x)$ is undefined.

note: just because $f''(x)=0$ or $f''(x)$ undefined does not mean that this is inflection pt.

EX 4 For this function, determine where it is increasing and decreasing, where it is concave up and down, find all max/min and inflection points. Use this information to sketch the graph.

$f(x) = 8x^{1/3} - x^{4/3} = 8\sqrt[3]{x} - x\sqrt[3]{x}$

(f is continuous everywhere)



important pts

$(2, 6\sqrt[3]{2})$ max

$(0, 0)$ vert. inflectn

$(-4, -12\sqrt[3]{4})$ inflectn pt

$f'(x) = \frac{8}{3}x^{-2/3} - \frac{4}{3}x^{1/3}$

$= \frac{8}{3\sqrt[3]{x^2}} - \frac{4}{3}\sqrt[3]{x}$

$= \frac{8-4x}{3\sqrt[3]{x^2}}$

$x \neq 0$ for f' $\Rightarrow$ $f'(0)$ is undefined

where is $f'(x) = 0$?

$(3\sqrt[3]{x^2})\frac{8-4x}{3\sqrt[3]{x^2}} = 0\,(3\sqrt[3]{x^2})$

$8 - 4x = 0$

$x = 2$

$f(0) = 8(0) - 0 = 0$

$f(2) = (8-2)\sqrt[3]{2} = 6\sqrt[3]{2} \approx 7.6$

$f(-4) = -12\sqrt[3]{4} \approx -19.05$

critical values: $x = 0, 2$

at $x=0$, not min or max

$f'(x)$: + on $(-\infty, 0)$ inc, + on $(0, 2)$ inc, − on $(2, \infty)$ dec; $f'(x) = \frac{8-4x}{3\sqrt[3]{x^2}}$

$f'(x) = \frac{8}{3}x^{-2/3} - \frac{4}{3}x^{1/3}$

$f''(x) = -\frac{16}{9}x^{-5/3} - \frac{4}{9}x^{-2/3} = \frac{-16}{9x\sqrt[3]{x^2}} - \frac{4}{9\sqrt[3]{x^2}}\left(\frac{x}{x}\right)$

note: $x \neq 0$

$= \boxed{\frac{-16-4x}{9x\sqrt[3]{x^2}}} = 0$ when $x = -4$

$f''(x)$: − on $(-\infty, -4)$ ∩, + on $(-4, 0)$ ∪, − on $(0, \infty)$ ∩

test:

$x = -8$: $\frac{-16+32}{9(-8)\sqrt[3]{64}} \rightarrow \frac{+}{-}$

$x = 1$: $\frac{-16-4}{9} < 0$

$x = -1$: $\frac{-16+4}{-9} = \frac{-12}{-9} > 0$

inflectn pts:

$(-4, -12\sqrt[3]{4})$, $(0, 0)$ vertical inf. pt.