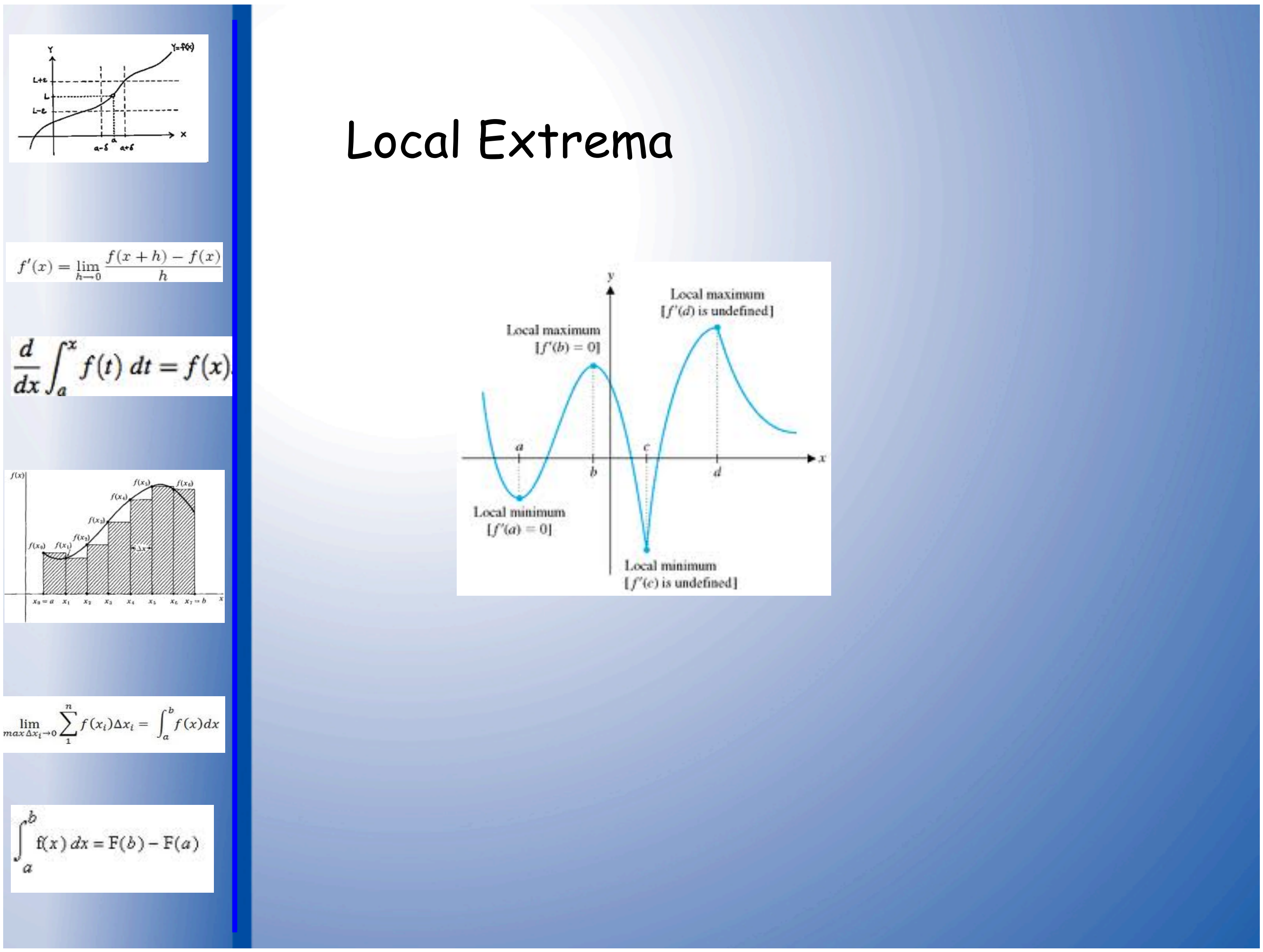
18B Local Extrema



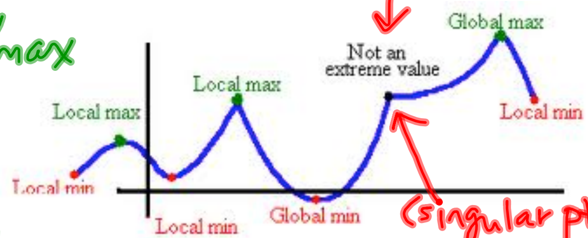
<u>Definition</u>

Let *S* be the domain of *f* such that *c* is an element of *S*.

Then,

1) *f(c)* is a **<u>local maximum</u>** value of *f* if there exists an interval *(a,b)* containing *c* such that *f(c)* is the maximum value of *f* on $(a,b)\cap S$.
2) *f(c)* is a **<u>local minimum</u>** value of *f* if there exists an interval *(a,b)* containing *c* such that *f(c)* is the minimum value of *f* on $(a,b)\cap S$.
3) f(c) is a **<u>local extreme value</u>** of *f* if it is either a local maximum or local minimum value.

local min/max pts are "peaks" & "valley" pts



(singular pt)

global max/min pts a.k.a. absolute max/min pts.

global max pt: pt on graph w/ highest y-value

How do we find the local extrema?

<u>First Derivative Test</u>

Let *f* be continuous on an open interval *(a,b*) that contains a critical *x-value*.

1) If *f'(x) > 0* for all *x* on *(a,c)* and *f'(x)<0* for all *x* on *(c,b),* then *f(c*) is a local maximum value.
2) If *f'(x) < 0* for all *x* on *(a,c)* and *f'(x)>0* for all *x* on *(c,b),* then *f(c*) is a local maximum value.
3) If f'(x) has the same sign on both sides of c, then *f(c)* not a maximum nor a minimum value.

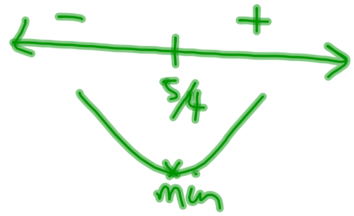
basically:
to find all min/max pts (local and global),
take first derivative,
① look for x-values that make it undefined
and
② look for x-values that make y' = 0

18B Local Extrema

EX 1 Determine local maximum and minimum points for $y = 2x^2 - 5x + 3$.

$y' = \boxed{4x - 5} = 0$ (no singular pts)

$x = 5/4$



$\boxed{\begin{array}{l} \text{no max pts.} \\ \text{min pt } (5/4, -1/8) \end{array}}$ global min

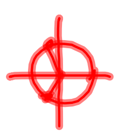
$y(5/4) = 2\left(\frac{5}{4}\right)^2 - 5\left(\frac{5}{4}\right) + 3$

$= \frac{25}{8} - \frac{25}{4} + 3$

$= \frac{-25}{8} + 3 = -1/8$

EX 2 Find all local maximum and minimum points for $f(x) = \frac{1}{2}x + sinx$ on $[0, 2\pi]$.

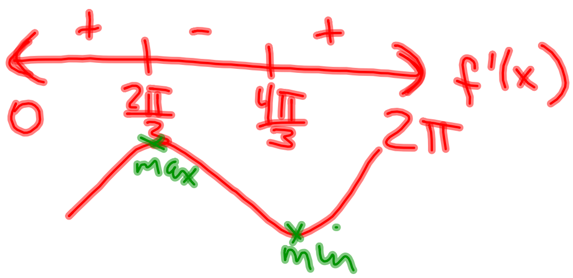
$f'(x) = \boxed{\frac{1}{2} + \cos x} = 0$ (no singular pts)



$\cos x = -1/2$

$x = 2\pi/3,\ 4\pi/3$

$\boxed{\begin{array}{l} \text{max } \left(\frac{2\pi}{3}, \frac{\pi}{3} + \frac{\sqrt{3}}{2}\right) \\ \text{min } \left(\frac{4\pi}{3}, \frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) \end{array}}$



test:

$x = \pi/6$, $(+) + (+)$

$x = \pi$, $\frac{1}{2} + -1$

$x = 3\pi/2$, $\frac{1}{2} + 0$

$f(x) = \frac{1}{2}x + \sin x$

$f\left(\frac{2\pi}{3}\right) = \frac{1}{2}\left(\frac{2\pi}{3}\right) + \sin(2\pi/3) = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$

$f\left(\frac{4\pi}{3}\right) = \frac{1}{2}\left(\frac{4\pi}{3}\right) + \sin(4\pi/3) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

Theorem: Second Derivative Test (this gives another way to confirm min/max pts.)

Let f' and f'' exist at every point on the interval (a,b) containing c and $f'(c) = 0$.

1) If $f''(c) < 0$, then $f(c)$ is a local maximum. max

2) If $f''(c) > 0$, the $f(c)$ is a local minimum. min

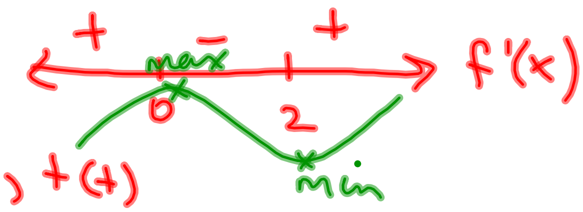
EX 3 Find all critical points for $f(x) = x^3 - 3x^2 + 1$.

(min/max pts)

$f'(x) = 3x^2 - 6x = 0$ (no singular pts)

$\boxed{3x(x-2)} = 0$

$x = 0, 2$



test: $x=-1$, $-(-)$ $x=3$, $+(+)$

$x=1$, $+(-)$

$f''(x) = 6x - 6$

$f''(0) = -6 < 0 \Rightarrow$ concave down at $x=0 \Rightarrow$ max

$f''(2) = 12-6 = 6 > 0 \Rightarrow$ concave up at $x=2 \Rightarrow$ min

critical pts	
$(0, 1)$	local max
$(2, -3)$	local min

$f(x) = x^3 - 3x^2 + 1$

$f(0) = 1$

$f(2) = 8 - 4(3) + 1 = -3$

EX 4 Find local and global extrema for $y = x^2 + \frac{1}{x^2}$ on $[-2, 2]$.

note: there's a VA at $x=0$ (we expect all derivatives to also be undefined at $x=0$)

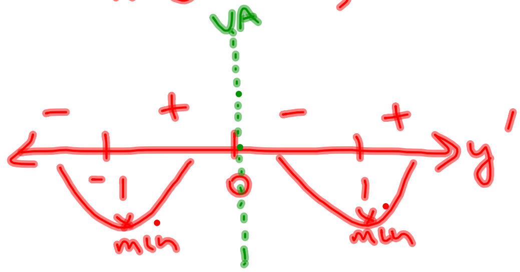
$y' = 2x + \frac{-2}{x^3} = 0$ (critical values: $x=0$)

$\boxed{\frac{2x^4-2}{x^3}} = 0$

$2x^4 - 2 = 0$

$x^4 = 1$

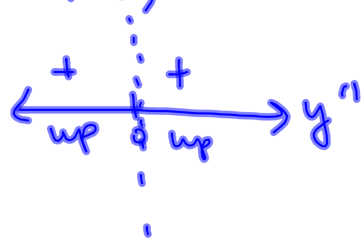
$x = \pm 1$



test: $x=-2$, $\frac{+}{-}$ $x = -1/2$, $\frac{-}{-}$ $x = 1/2$, $\frac{-}{+}$ $x = 1000$, $\frac{+}{+}$

$y'' = 2 + \frac{-2(-3)}{x^4}$ (problem at $x=0$)

$= \frac{2x^4+6}{x^4} > 0$ always



on $[-2, 2]$

min $(-1, 2)$

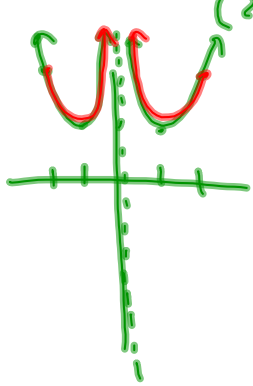
min $(1, 2)$

$y = x^2 + \frac{1}{x^2}$

$y(\pm 1) = 1 + 1 = 2$

$y(\pm 2) = 4 + \frac{1}{4} = \frac{17}{4}$

endpts $(-2, 4\frac{1}{4})$
$(2, 4\frac{1}{4})$

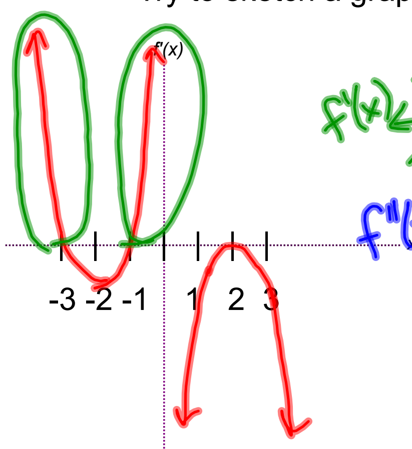


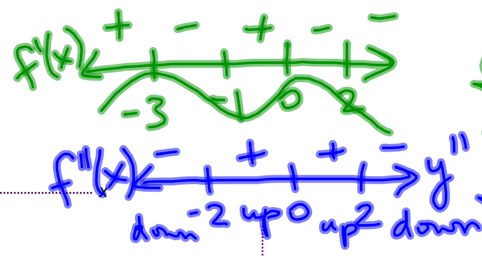
$\Rightarrow$ no global max (because graph goes up to ∞)

global min pts $(\pm 1, 2)$

EX 5 Let f be continuous such that *f′* has the following graph.

Try to sketch a graph of *f(x)* and answer these questions.





a) Where is f increasing?

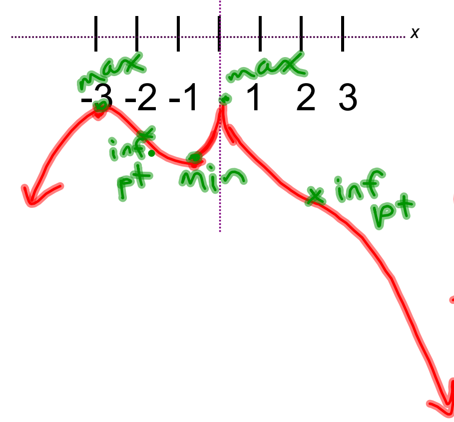
b) Where is f decreasing?

c) Where is f concave up?

d) Where is f concave down?

e) Where are inflections points?

f) Where are local max/min values?



min/max pts:

at x = -3 (max)

x = -1 (min)

(singular) x = 0 (max)

inflection pts:

at x = -2

and x = 2

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