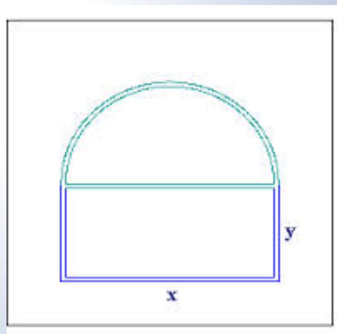
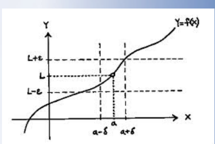
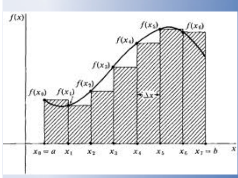
Optimization Problems





$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx} \int_a^x f(t)\,dt = f(x)$$

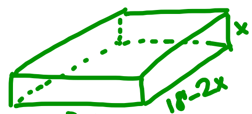


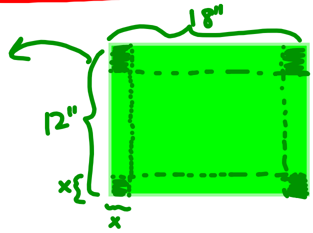
$$\lim_{max\,\Delta x_i \to 0} \sum_1^n f(x_i)\Delta x_i = \int_a^b f(x)dx$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

EX 1 An open box is made from a 12" by 18" rectangular piece of cardboard by cutting equal squares from each corner and turning up the sides. Find the volume of the largest box that can be made in this manner.

$V = ?$





cut out corner square pieces and fold up to create box

goal: Maximize volume

$0 < x < 6$

$V = lwh$

$V = (12-2x)(18-2x)x$

$V = (216 - 24x - 36x + 4x^2)x$

$V = 4x^3 - 60x^2 + 216x$

$V' = 12x^2 - 120x + 216 = 0$

$12(x^2 - 10x + 18) = 0$

$\cancel{12(x \quad)(x \quad) = 0}$

$x = \dfrac{10 \pm \sqrt{10^2 - 4(18)}}{2} = \dfrac{10 \pm \sqrt{28}}{2}$

$= \dfrac{10 \pm 2\sqrt{7}}{2} = 5 \pm \sqrt{7}$

note: $5 + \sqrt{7} > 6 \Rightarrow x \neq 5 + \sqrt{7}$

Check: $x = 5 - \sqrt{7}$, but is this a max or a min?

take 2nd derivative: $V' = 12x^2 - 120x + 216$

$V'' = 24x - 120 \Rightarrow V''(5-\sqrt{7}) = 24(5-\sqrt{7}) - 120$

$= 120 - 24\sqrt{7} - 120 = -24\sqrt{7} < 0$

$\Rightarrow$ $y = V(x)$ graph is concave down at $x = 5 - \sqrt{7}$, so it's a max pt!

$\Rightarrow$ max volume is

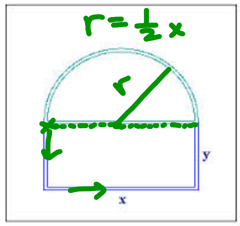
$V(5-\sqrt{7}) = 4(5-\sqrt{7})^3 - 60(5-\sqrt{7})^2 + 216(5-\sqrt{7})$

$\approx \boxed{228.16 \text{ in}^3}$

EX 2 A Norman window is constructed by adjoining a semicircle to the top of an ordinary rectangular window. Find the dimensions of the window of maximum area if the total outer perimeter is 18 meters.

goal
$x, y, r = ?$

maximize area



$$A = xy + \frac{\pi r^2}{2}$$

(note: I'll keep x as the input var.)

$r = \frac{1}{2}x$, $18 = x + 2y + \pi r$

$$18 = x + 2y + \pi\left(\frac{x}{2}\right)$$

$$18 - x - \frac{\pi}{2}x = 2y$$

$$y = 9 - \frac{1}{2}x - \frac{\pi}{4}x$$

$$\Rightarrow A = A(x) = x\left(9 - \frac{1}{2}x - \frac{\pi}{4}x\right) + \frac{\pi}{2}\left(\frac{1}{2}x\right)^2$$

$$A(x) = 9x - \frac{1}{2}x^2 - \frac{\pi}{4}x^2 + \frac{\pi}{8}x^2$$

$$\cap \quad A(x) = -\frac{1}{2}x^2 - \frac{\pi}{8}x^2 + 9x$$

note: this is concave down parabola $\Rightarrow$ we'll get max pt for vertex.

$$A'(x) = -x - \frac{\pi}{4}x + 9 = 0$$

$$x\left(-1 - \frac{\pi}{4}\right) = -9$$

$$x = \frac{9}{1 + \pi/4} = \frac{36}{4 + \pi} \approx \boxed{5.04\text{m}}$$

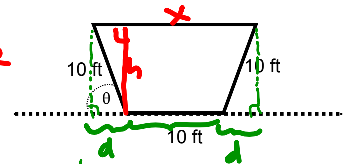
$$r = \frac{1}{2}x = \frac{18}{4+\pi} \approx \boxed{2.52\text{m}}$$

$$y = 9 - \left(\frac{1}{2} + \frac{\pi}{4}\right)\left(\frac{36}{4+\pi}\right) \approx \boxed{2.52\text{m}}$$

Strategy for Optimization Problems

1. write down goal
2. Find a function to maximize/minimize.
3. Need our fn to be a fn of only one variable $\Rightarrow$ use info given to rewrite all other vars. in terms of one of the input vars.
4. Take derivative of the fn to find min/max pts (note: consider domain for the context of the problem)
5. verify that we found the pt we want
6. answer the question

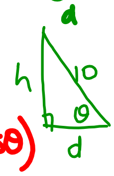
EX 3 The cross-sections of an irrigation canal are isosceles trapezoids with lengths as shown. Determine the angle of elevation of the sides so that the area of the cross sections is maximum.



maximize area

$\theta = ?$ $0 < \theta < \pi/2$

$A = \frac{h}{2}(10 + x)$



$\sin\theta = \frac{h}{10}$

$h = 10\sin\theta$

$A = A(\theta) = \frac{10\sin\theta}{2}(10 + 10 + 20\cos\theta)$

$\cos\theta = \frac{d}{10}$

$d = 10\cos\theta$

$x = 10 + 2d$

$A(\theta) = 5\sin\theta(20 + 20\cos\theta)$

$= 100\sin\theta + 100\sin\theta\cos\theta$

$A'(\theta) = 100\cos\theta +$

$100(\cos\theta\cos\theta + \sin\theta(-\sin\theta))$

$= 100\cos\theta + 100\cos^2\theta - 100\sin^2\theta$

$100(\cos\theta + \cos^2\theta - (1 - \cos^2\theta)) = 0$

$100(\cos\theta + 2\cos^2\theta - 1) = 0$

$100(2\cos^2\theta + \cos\theta - 1) = 0$

$100(2\cos\theta - 1)(\cos\theta + 1) = 0$

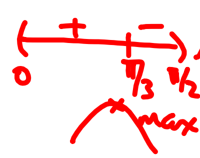
$2\cos\theta - 1 = 0$ or $\cos\theta + 1 = 0$

$\cos\theta = 1/2$ $\cos\theta = -1$

$\theta = \pi/3$ $\cancel{\theta = \pi}$

check that this gives max (not min) area.

$A'(\theta)$ sign line



$A'(\theta) = 100(2\cos\theta - 1)(\cos\theta + 1)$

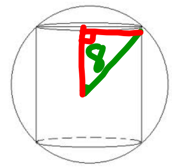
$\boxed{\theta = \pi/3}$

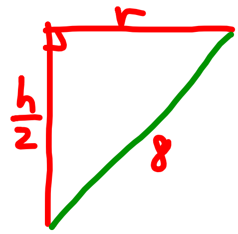
EX 4 Find the volume of the largest right circular cylinder that can be inscribed in a sphere of radius 8m.

maximize volume

$V = ?$

$V = \pi r^2 h$





want r in terms of h:
use Pythagorean Thm

$r^2 + \left(\frac{h}{2}\right)^2 = 64$

$r^2 = 64 - \frac{1}{4}h^2$

$V = V(h) = \pi\left(64 - \frac{1}{4}h^2\right)h$

$V(h) = -\frac{\pi}{4}h^3 + 64\pi h$

$V'(h) = -\frac{3\pi}{4}h^2 + 64\pi = 0$

$-\frac{3\pi}{4}h^2 = -64\pi$

$h^2 = \frac{256}{3}$

$\Rightarrow h = \frac{\pm 16}{\sqrt{3}}$ $h > 0$ must be true

$h = \frac{16}{\sqrt{3}}$

check that this is actually a max (not min)

$V'' = -\frac{3\pi}{2}h < 0$ for $h > 0 \Rightarrow$ concave down at $h = \frac{16}{\sqrt{3}}$

$\Rightarrow$ it's a max

$\Rightarrow$ max Volume

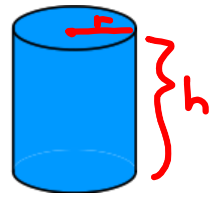
$V = \pi r^2 h = \pi\left(64 - \frac{1}{4}\left(\frac{256}{3}\right)\right)\left(\frac{16}{\sqrt{3}}\right)$

$\approx \boxed{1238.22\ m^3}$

EX 5 A right circular cylinder is to be designed to hold a liter of water. Find the dimensions of such a cylinder which uses the least amount of material in its construction.

minimize surface area

$r, h = ?$



1 l of water $= 1000\text{ cm}^3$

$SA = 2\pi r^2 + 2\pi r h$

Volume $= 1000\text{ cm}^3$

$V = 1000 = \pi r^2 h$

$h = \dfrac{1000}{\pi r^2}$

$SA = 2\pi r^2 + 2\pi r\left(\dfrac{1000}{\pi r^2}\right)$

$SA = 2\pi r^2 + \dfrac{2000}{r}$

$SA' = 4\pi r - \dfrac{2000}{r^2} = 0$

$4\pi r^3 - 2000 = 0$

$r^3 = \dfrac{500}{\pi} \Rightarrow r = \sqrt[3]{\dfrac{500}{\pi}}$

check SA'':

$SA'' = 4\pi + \dfrac{2000}{r^3} > 0$ for $r > 0$

$\Rightarrow r = \sqrt[3]{\dfrac{500}{\pi}}$ is at a min

$r = \sqrt[3]{\dfrac{500}{\pi}} = \left(\dfrac{500}{\pi}\right)^{1/3}$

$h = \dfrac{1000}{\pi\left(\dfrac{500}{\pi}\right)^{2/3}} = \dfrac{2(500)}{\pi(\pi^{-2/3})(500^{2/3})}$

$= \dfrac{2(500^{1/3})}{\pi^{1/3}} = 2\left(\dfrac{500}{\pi}\right)^{1/3} = 2\sqrt[3]{\dfrac{500}{\pi}}$

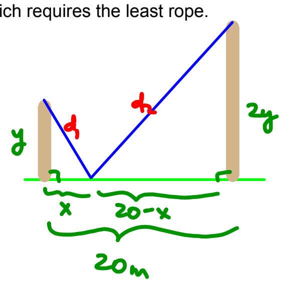
(h is twice the radius!)

$\boxed{r \simeq 5.42\text{ cm},\ h \simeq 10.84\text{ cm}}$

EX 6 Two vertical poles which are 20 m apart are secured by a rope going from the top of the first pole to a point on the ground between the poles and then to the top of the second pole. The second pole is twice as tall as the first pole.
Find the position of attachment which requires the least rope.

$x = ?$ so

$d_1 + d_2$ is minimum



$d_1 = \sqrt{x^2+y^2}$

$d_2 = \sqrt{4y^2+(20-x)^2}$

Note: We will assume y is fixed (and not the variable)

fn to optimize:

$$f(x) = \sqrt{x^2+y^2} + \sqrt{4y^2+(20-x)^2}$$

domain: $0 < x < 20$

$$f'(x) = \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2}} + \frac{\cancel{2}(20-x)(-1)}{\cancel{2}\sqrt{4y^2+(20-x)^2}}$$

$$= \frac{x}{\sqrt{x^2+y^2}} + \frac{x-20}{\sqrt{4y^2+(20-x)^2}} = 0$$

solve for x:

$$x\sqrt{4y^2+(20-x)^2} + (x-20)\sqrt{x^2+y^2} = 0$$

$$\left(x\sqrt{4y^2+(20-x)^2}\right)^2 = \left((20-x)\sqrt{x^2+y^2}\right)^2$$

$$x^2\left(4y^2+(20-x)^2\right) = (20-x)^2(x^2+y^2)$$

$$4x^2y^2 + \cancel{x^2(20-x)^2} = \cancel{x^2(20-x)^2} + y^2(20-x)^2$$

(can divide both sides by y^2, since we know $y \neq 0$)

$$4x^2y^2 = 400y^2 - 40xy^2 + x^2y^2$$

$$3x^2y^2 + 40y^2x - 400y^2 = 0$$

$$x = \frac{-40y^2 \pm \sqrt{1600y^4 - 4(3y^2)(-400y^2)}}{2(3y^2)}$$

$$x = \frac{-40y^2 \pm \sqrt{1600y^4(1+3)}}{6y^2}$$

$$x = \frac{-40y^2 \pm 80y^2}{6y^2} = \frac{-20 \pm 40}{3}$$

$$x = \cancel{-20} \text{ or } \frac{20}{3}$$

$f'(x)$: 0 $-$ $\frac{20}{3}$ $+$ — min

For least rope, $x = \boxed{\frac{20}{3}\text{ m}}$

Summary

- look for key words (least, most, biggest, maximum, etc.)