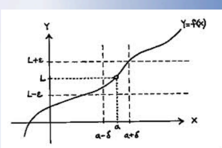
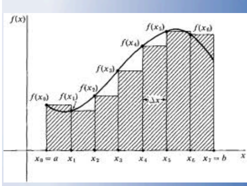


$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

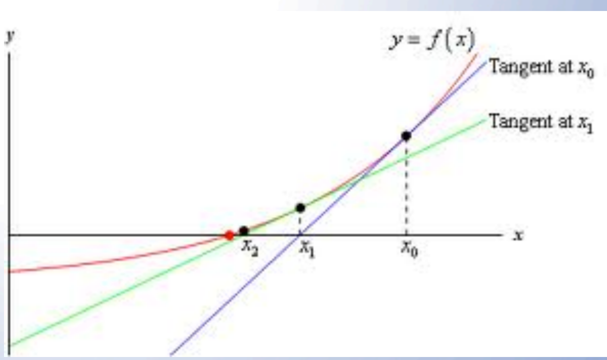
$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$



$$\lim_{max\, \Delta x_i \to 0} \sum_1^n f(x_i) \Delta x_i = \int_a^b f(x) dx$$

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Solving Equations Numerically



Three numeric methods for solving an equation numerically:

① Bisection Method

② Newton's Method

③ Fixed-point Method

① Bisection Method Algorithm

Let *f(x)* be a continuous function and let a_1 and b_1 be numbers satisfying $a_1<b_1$ and $f(a_1)\cdot f(b_1)<0$.

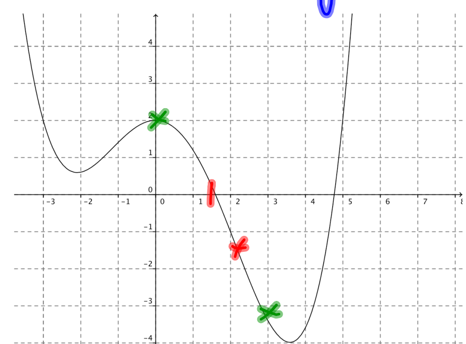
Let *E* denote the desired bound for the error $|r-m_n|$.

Repeat steps *1* to *5* for *n=1,2,...* until $h_n < E$

Pros: always works

Cons: converges slowly

1. Calculate $m_n = \dfrac{a_m + b_m}{2}$
2. Calculate $f(m_n)$ and if $f(m_n)=0$ then STOP.
3. Calculate $h_n = \left|\dfrac{b_n - a_n}{2}\right|$ (for error testing).
4. If $f(a_n)\cdot f(m_n)<0,$ then set $a_{n+1}=a_n$ and $b_{n+1}=m_n$.
5. If $f(a_n)\cdot f(m_n)>0,$ then set $a_{n+1}=m_n$ and $b_{n+1}=b_n$.



$y(0)>0$ if y is cont,
$y(3)<0$

$\Rightarrow$ there must be an x value where the curve crosses x-axis, i.e. $y=0$.

$y(1.5)>0$

$\Rightarrow$ there's a zero in $x\in(1.5,3)$

EX 1: Approximate the real root to 2 decimal places. $f(x) = x^4 + 5x^3 + 1$ *on* $[-1, 0]$

(use Bisection Method)

note: $f(-1) = 1 - 5 + 1 < 0$

$f(0) = 1 > 0$

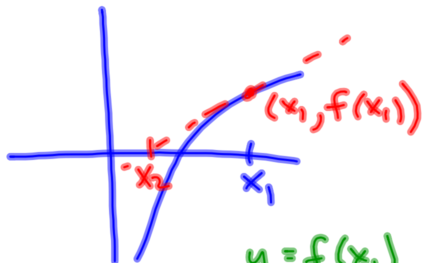
$\Longrightarrow$ there $\underline{\text{is}}$ an xvalue in $(-1, 0)$ such that $f(x) = 0$.

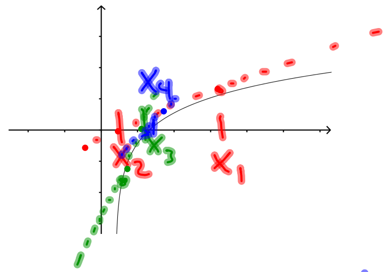
counter	left xvalue	right xvalue	midpt xvalue			
n	a_n	b_n	m_n	$f(a_n)$	$f(b_n)$	$f(m_n)$
1	-1	0	-0.5	-3	1	0.4375 replace b_n
2	-1	-0.5	-0.75	-3	0.4375	-0.7929688 replace a_n
3	-0.75	-0.5	-0.625	-0.7929688	0.4375	-0.0681152 replace a_n
4	-0.625	-0.5	-0.5625	-0.0681152	0.4375	0.21022034 replace b_n
5	-0.625	-0.5625	-0.59375	-0.0681152	0.21022034	0.07768345 replace b_n
6	-0.625	-0.59375	-0.609375	-0.0681152	0.07768345	0.00647169 ≈ 0 (up to 2 decimal places)

② Newton's Method Algorithm

Let $f(x)$ be a differentiable function and let x_1 be an initial approximation to the root, r of $f(x) = 0$. Let E denote a bound for the error $|r - x_n|$. Repeat the following step for $n = 1, 2, \ldots$ until $|x_{n+1} - x_n| < E$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$





Pros: fast
elegant

Cons:
it doesn't always converge (dependent on initial guess)

$y_1 = f(x_1)$

want tangent line thru (x_1, y_1) and slope of $f'(x_1)$

$$y - y_1 = f'(x_1)(x - x_1)$$

$$y = y_1 + f'(x_1)x - f'(x_1)x_1$$

we define x_2 to be x-value where tangent line has $y = 0$

plug in $y = 0$

$$0 = y_1 + f'(x_1)x - f'(x_1)x_1$$

$$f'(x_1)x_1 - y_1 = f'(x_1)x$$

$$x = \frac{f'(x_1)x_1 - y_1}{f'(x_1)}$$

$$x = \frac{f'(x_1)x_1}{f'(x_1)} - \frac{y_1}{f'(x_1)} = \boxed{x_1 - \frac{y_1}{f'(x_1)} = x_2}$$

Newton's method formula $\Rightarrow$ $\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$ (given x_1)

EX 2 Use Newton's method to approximate a root of $7x^3+2x-5=0$ to 5 decimal places.

$$f(x)=7x^3+2x-5$$

$$f'(x)=21x^2+2$$

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}=x_n-\frac{7x_n^3+2x_n-5}{21x_n^2+2}$$

$$=\frac{21x_n^3+\cancel{2x_n}-7x_n^3-\cancel{2x_n}+5}{21x_n^2+2}$$

$$\boxed{x_{n+1}=\frac{14x_n^3+5}{21x_n^2+2}}$$

n	x_n
1	1
2	0.8260869
3	0.78944826
4	0.7879276
5	0.7879250

$$x_2=\frac{14x_1^3+5}{21x_1^2+2}=\frac{14(1^3)+5}{21(1^2)+2}$$

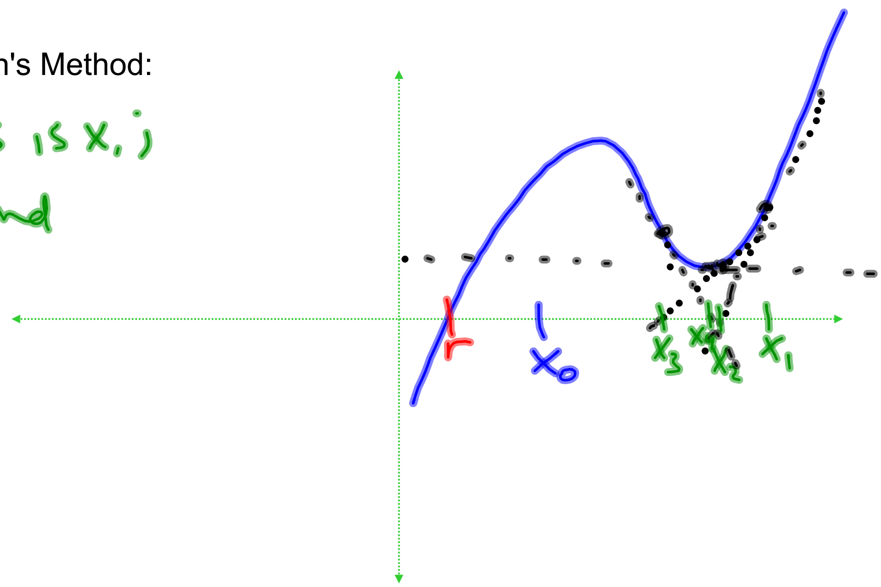
$$x_3=\frac{14(0.8260869)^3+5}{21(0.8260869)^2+2}$$

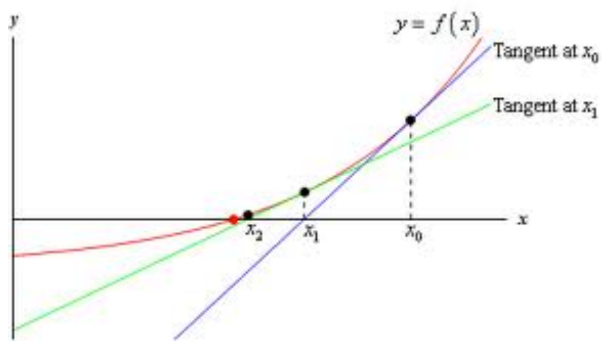
$$x_4=\frac{14(0.78\ldots)^3+5}{21(0.78\ldots)^2+2}$$

$\Rightarrow$ answer is $\simeq 0.78792$

Warning on Newton's Method:

original guess is x_0;
looking to find
r




y
$y = f(x)$
Tangent at x_0
Tangent at x_1
x
x_2
x_1
x_0