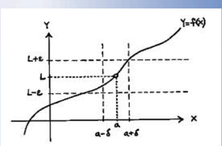
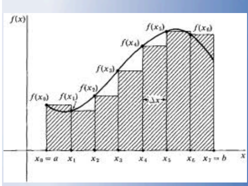


$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

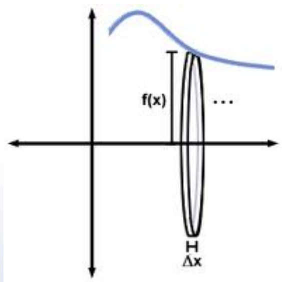
$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x)$$

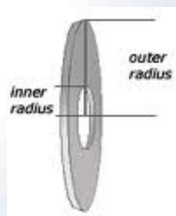


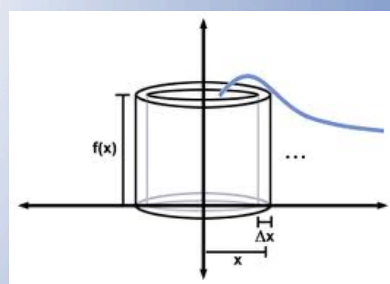
$$\lim_{max\, \Delta x_i \to 0} \sum_{1}^{n} f(x_i) \Delta x_i = \int_a^b f(x) dx$$

$$\int_a^b f(x)\, dx = F(b) - F(a)$$

Volume of Solids

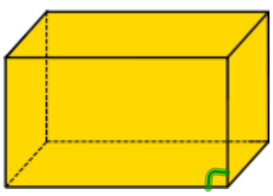


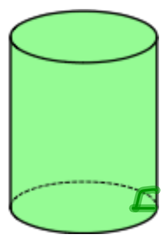




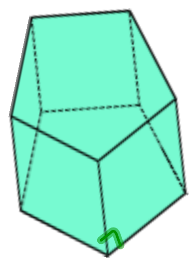
30B Volume Solids

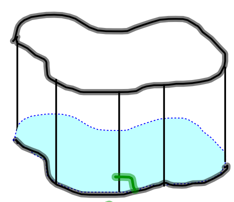
The volume of a solid right prism or cylinder is the area of the base times the height.





right circular cylinder



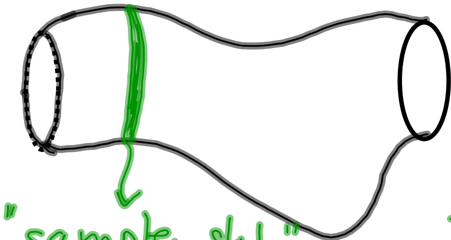


cylinder

30B Volume Solids

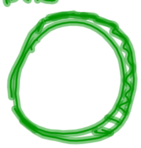
Disk Method : one method to find the volume of a solid (of revolution)

How would we find the volume of a shape like this?



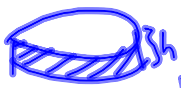
Cut up into pieces (slabs)

"sample slab" is a disk (like a penny)



to find the total volume: add up all sample slabs (add up volumes of all the disks/pennies)

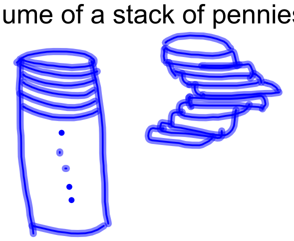
Volume of one penny:



it's a right circular cylinder

$V_{penny} = A_{base} \cdot h = \pi r^2 h$

Volume of a stack of pennies:



$$\text{total volume} = \sum_{\text{all coins}} V_{coin}$$

Formula $\displaystyle \lim_{n\to\infty} \sum_{i=1}^{n} A_{\substack{base\\coin}} \cdot h_{coin} = \lim_{n\to\infty} \sum_{i=1}^{n} (\pi r_i^2)\, dx \text{ (or } dy)$

$$= \int_a^b \pi r^2 \, dx \text{ (or } dy)$$

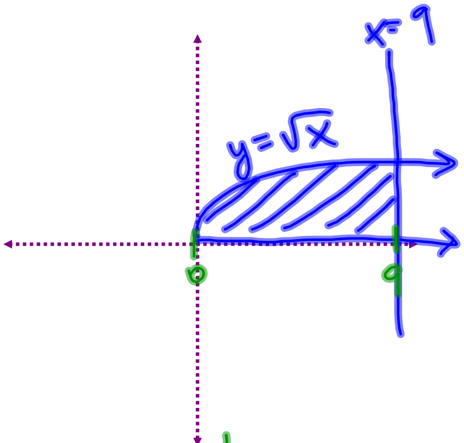
disk method (generic)

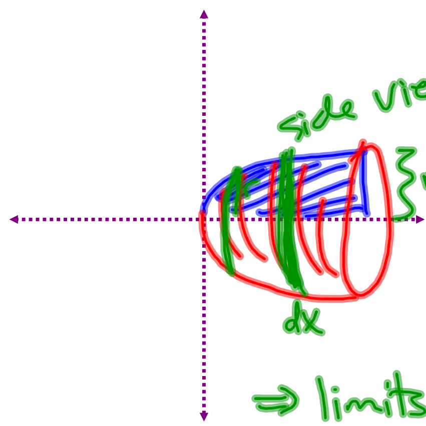
r = radius of a sample disk (fn of x or fn of y)

a = smallest x-value if dx, or y-value if dy

b = largest " " " " " "

EX 1 Find the volume of the solid of revolution obtained by revolving the region bounded by $y=\sqrt{x}$, the x-axis and the line $x=9$ about the x-axis.





side view of sample disk

r is a vert. distance

⇒ limits of integration are x values and r is a fn of x

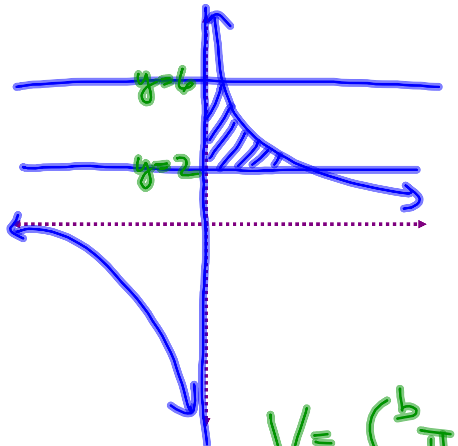
$$V=\int_a^b \pi r^2\,dx$$
$$=\int_0^9 \pi r^2\,dx$$
$$=\int_0^9 \pi(\sqrt{x})^2\,dx$$
$$=\pi\int_0^9 x\,dx$$
$$=\pi\left(\frac{x^2}{2}\right)\Big|_0^9=\pi\left(\frac{81}{2}-0\right)$$
$$=\boxed{\frac{81\pi}{2}}\text{ units}^3$$

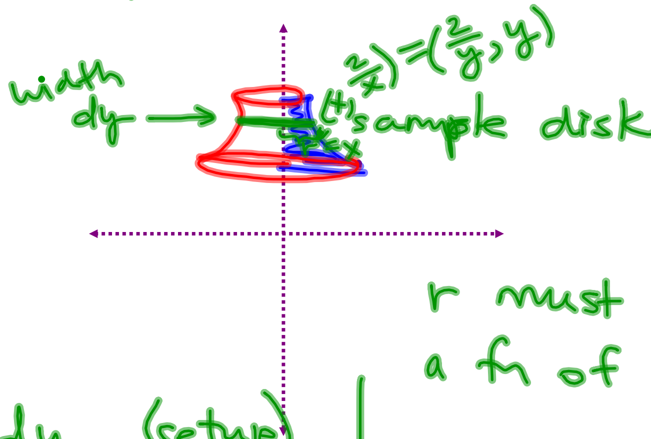
★ important: r is always measured from axis of rotation

EX 2 Find the volume of the solid generated by revolving the region enclosed by $x=\frac{2}{y}$, $y=2$, $y=6$, and $x=0$ about the y-axis.

$y=\frac{2}{x}$

disk $V=\int_a^b \pi r^2\,dx$ or dy





r must be a fn of y

r = horiz. dist from y-axis to curve
$y=\frac{2}{x}$ or $\boxed{x=\frac{2}{y}}$

$$V=\int_a^b \pi r^2\,dy \quad \text{(setup)}$$

$$=\int_2^6 \pi r^2\,dy$$

$$=\pi\int_2^6\left(\frac{2}{y}\right)^2 dy$$

$$=4\pi\int_2^6 y^{-2}\,dy$$

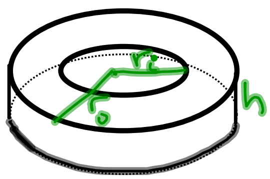
$$=4\pi\left(\frac{y^{-1}}{-1}\right)\Big|_2^6=-4\pi\left(\frac{1}{y}\Big|_2^6\right)$$

$$=-4\pi\left(\frac{1}{6}-\frac{1}{2}\right)$$

$$=-4\pi\left(\frac{-1}{3}\right)=\boxed{\frac{4\pi}{3}}\text{ units}^3$$

Washer Method

How would we find the volume of a washer?



$V_{washer} =$ area of base $\cdot$ ht

r_i = inner radius

r_o = outer radius

<u>area of base:</u>

$A_{base} = \pi r_o^2 - \pi r_i^2$

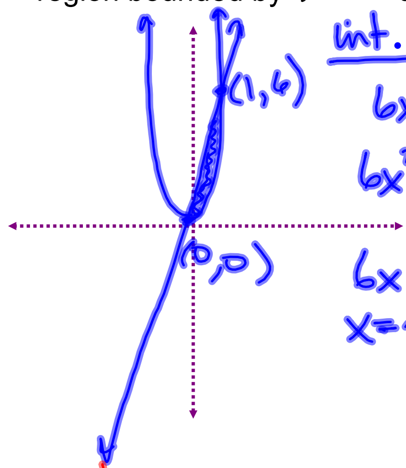
WARNING: $A_{base} \neq \pi(r_o - r_i)^2$

$\Rightarrow V_{washer} = \pi(r_o^2 - r_i^2) \cdot h$

Volume of solid with a cylindrical hole (ie. volume of solid of revolution where each cross section is a washer instead of disk).

$V = \pi \int_a^b (r_o^2 - r_i^2)\, dx$ or dy

EX 3 Find the volume of the solid generated by revolving about the x-axis the region bounded by ① $y = 6x$ and ② $y = 6x^2$.



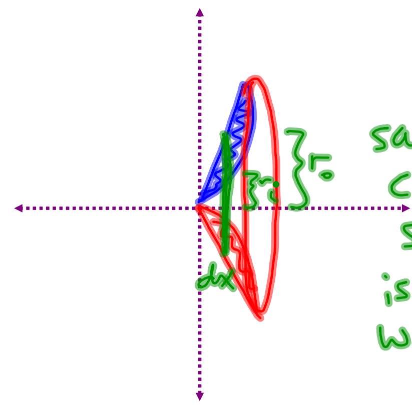
int. pts

$6x = 6x^2$

$6x^2 - 6x = 0$

$6x(x-1) = 0$

$x = 0, 1$



sample cross section is a washer!

$$V = \pi \int_a^b (r_o^2 - r_i^2)\,dx$$

$$= \pi \int_0^1 (r_o^2 - r_i^2)\,dx$$

$$= \pi \int_0^1 \left((6x)^2 - (6x^2)^2\right) dx$$

r_o = vert dist. from x-axis to line $y = 6x$

r_i = vert dist. from x-axis to curve $y = 6x^2$

$$= \pi \int_0^1 (36x^2 - 36x^4)\,dx$$

$$= \pi \left(\frac{36x^3}{3} - \frac{36x^5}{5} \right)\Big|_0^1$$

$$= \pi \left(\left(12(1^3) - \frac{36}{5}(1^5)\right) - (0 - 0) \right)$$

$$= \pi \left(12 - \frac{36}{5}\right) = \boxed{\frac{24\pi}{5}} \text{ units}^3$$

washer/disk method

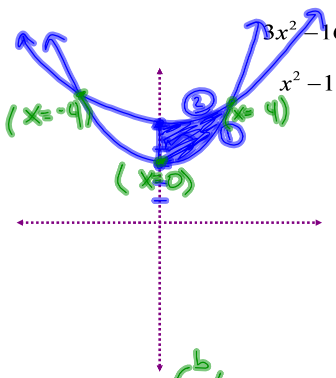
① if rotate about horiz line, each disk/washer will have dx width

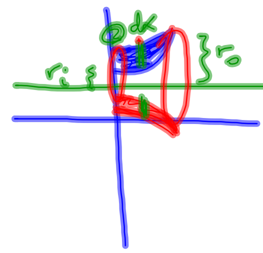
② if rotate about a vert. line, each disk/washer will have dy width

EX 4 Find the volume of the solid generated by revolving about the line $y = 2$ the region in the first quadrant bounded by these parabolas and the y-axis. (Hint: Always measure radius from the axis of revolution.)

$3x^2 - 16y + 48 = 0$ ① $\quad y = \frac{3}{16}x^2 + 3$

$x^2 - 16y + 80 = 0$ ② $\quad y = \frac{1}{16}x^2 + 5$





each cross section is a washer

$$V = \pi\int_a^b (r_o^2 - r_i^2)\,dx$$

$$= \pi\int_0^4 (r_o^2 - r_i^2)\,dx$$

$$\boxed{= \pi\int_0^4 \left(\tfrac{1}{16}x^2+5-2\right)^2 - \left(\tfrac{3}{16}x^2+3-2\right)^2 dx}$$

find intersectn pt:

① $y = \frac{3}{16}x^2 + 3$ ② $y = \frac{1}{16}x^2 + 5$

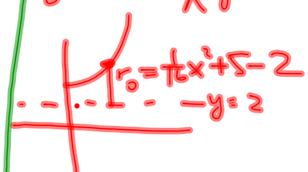
$$\tfrac{1}{16}x^2 + 5 = \tfrac{3}{16}x^2 + 3$$

$$2 = \tfrac{1}{8}x^2$$

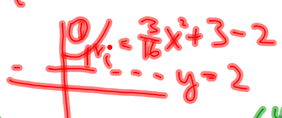
$$16 = x^2$$

$$\Rightarrow x = \pm 4$$

r_o = vert. dist. from $y=2$ to $(x, \frac{1}{16}x^2+5)$



r_i = vert. dist from $y=2$ to $(x, \frac{3}{16}x^2+3)$



$$V = \pi\int_0^4 \left(\tfrac{1}{16}x^2+3\right)^2 - \left(\tfrac{3}{16}x^2+1\right)^2 dx$$

$$= \pi\int_0^4 \left(\left(\tfrac{1}{256}x^4 + \tfrac{3}{8}x^2 + 9\right) - \left(\tfrac{9}{256}x^4 + \tfrac{3}{8}x^2 + 1\right)\right) dx$$

$$= \pi\int_0^4 \left(\tfrac{-1}{32}x^4 + 8\right) dx$$

$$= \pi\left(\tfrac{-1}{32}\left(\tfrac{x^5}{5}\right) + 8x\right)\Big|_0^4$$

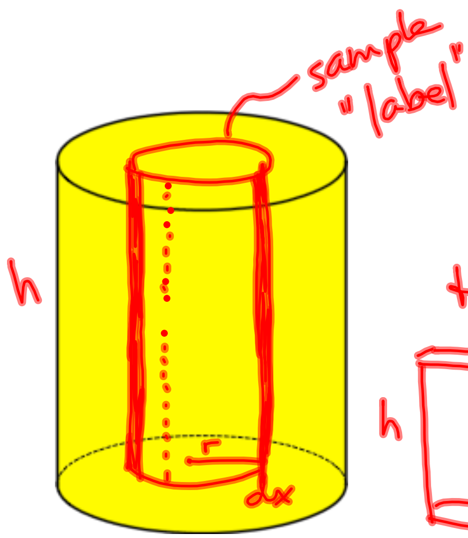
$$= \pi\left[\left(\tfrac{-1}{160}(4^5) + 8(4)\right) - (0+0)\right]$$

$$= \pi\left(\tfrac{-1}{10}(64) + 32\right)$$

$$= \boxed{25.6\pi} \text{ units}^3$$

Shell Method 2nd way to find volume of solid of revolution.

How would we find the volume of a label we peel off a can?



Volume of sample label

cut label down an edge + unfold/unravel

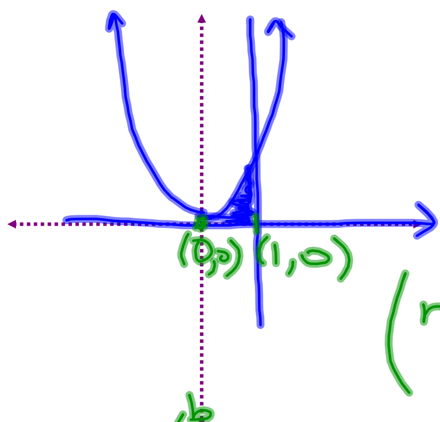
$V_{label} = A_{rect} \cdot dx = (2\pi r)h\ dx$

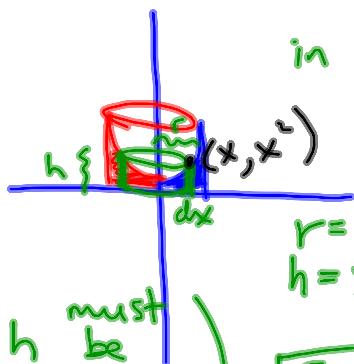
$$\boxed{V_{solid} = \int_a^b 2\pi r h\ dx \text{ (or } dy)}$$

generic conceptual formula for shell method

EX 5 Find the volume of the solid generated when the region bounded by these three equations is revolved about the y-axis.

$\boxed{y = x^2}$ $\quad x = 1 \quad$ $y = 0$





in green: sample shell

$r = x$

$h = x^2$

(r + h must be fns of x)

$$V = 2\pi \int_a^b rh\,dx$$

$$= 2\pi \int_0^1 rh\,dx$$

$$= 2\pi \int_0^1 x(x^2)\,dx$$

$$= 2\pi \left(\frac{x^4}{4} \Big|_0^1 \right) = \frac{2\pi}{4}(1-0)$$

$$= \frac{\pi}{2} \text{ units}^3$$

For shell method

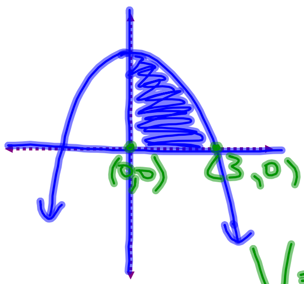
① if axis of rotatn is horizontal, then width will be dy

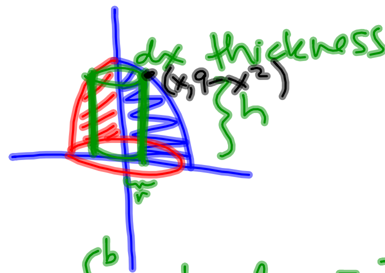
② if axis of rotatn is vertical, then width will be dx

EX 6 Find the volume of the solid generated when the region bounded by these equations is revolved about the y-axis in two ways.

$y = 9 - x^2 \;, x \geq 0 \qquad x = 0 \qquad y = 0$

① Shell method





$\Rightarrow r = x$

$h = 9 - x^2$

$V = 2\pi \int_a^b rh\,dx = 2\pi \int_0^3 rh\,dx$

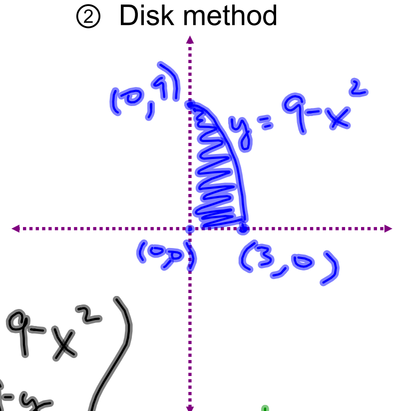
$V = 2\pi \int_0^3 x(9-x^2)\,dx$

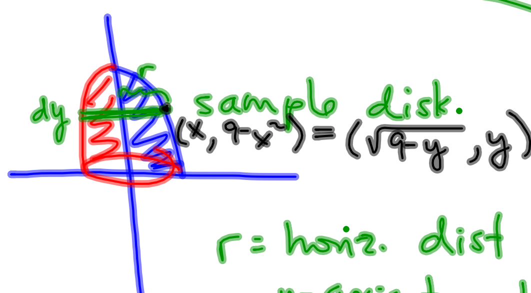
$= 2\pi \left(\frac{9x^2}{2} - \frac{x^4}{4} \right)\Big|_0^3$

$= 2\pi \left(\left(\frac{81}{2} - \frac{81}{4} \right) - 0 \right)$

$= 2\pi(81)\left(\frac{1}{4}\right) = \boxed{\frac{81\pi}{2}}$ units^3

② Disk method





$y = 9 - x^2$

$x^2 = 9 - y$

$x = \pm\sqrt{9-y}$

in QI

$x = \sqrt{9-y}$

r = horiz. dist from y-axis to pt on parabola

$V = \pi \int_a^b r^2\,dy$

$= \pi \int_0^9 r^2\,dy = \pi \int_0^9 \left(\sqrt{9-y}\right)^2 dy$

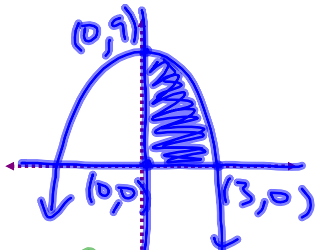
$= \pi \int_0^9 (9-y)\,dy$

$= \pi \left(9y - \frac{y^2}{2} \right)\Big|_0^9 = \pi\left(\left(81 - \frac{81}{2} \right) - 0 \right)$

$= \pi\left(\frac{81}{2}\right) = \boxed{\frac{81\pi}{2}}$ units^3

EX 7 Find the volume of the solid generated when the region in quadrant 1 bounded by these equations is revolved about the line $x = 3$.

$y = 9 - x^2,\ x \geq 0 \qquad x = 0 \qquad y = 0$



shell method

h = vert dist from x-axis to pt $(x, 9-x^2)$

$\Rightarrow h = 9 - x^2$

r = horiz dist from line x = 3 to pt $(x, 9-x^2)$

$r = 3 - x$

$$V = 2\pi \int_a^b r h \, dx$$

$$= 2\pi \int_0^3 r h \, dx = 2\pi \int_0^3 (3-x)(9-x^2)\,dx$$

$$= 2\pi \int_0^3 (27 - 9x - 3x^2 + x^3)\,dx$$

$$= 2\pi \left(27x - \frac{9x^2}{2} - \frac{3x^3}{3} + \frac{x^4}{4}\right)\Big|_0^3$$

$$= 2\pi \left(\left(27(3) - \frac{9(9)}{2} - 3^3 + \frac{3^4}{4}\right) - 0\right)$$

$$= 2\pi \left(81 - \frac{81}{2} - 27 + \frac{81}{4}\right)$$

$$= 2\pi \left(\frac{81}{2} + \frac{81}{4} - 27\right)$$

$$= 2\pi \left(\frac{3}{4}(81) - 27\right)$$

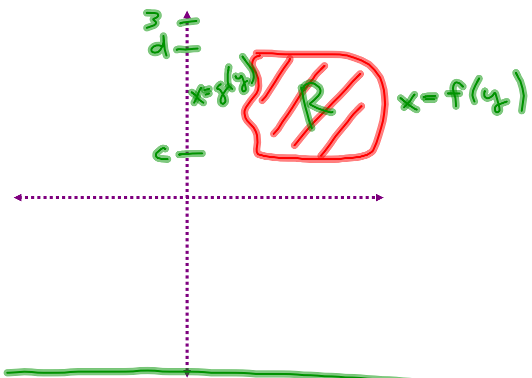
$$= 2\pi \left(\frac{9(27) - 4(27)}{4}\right) = \frac{\pi}{2}(5(27)) = \boxed{\frac{135\pi}{2}} \text{ units}^3$$

EX 8 A region, *R* is shown below. Set up an integral for the volume obtained by revolving *R* about the given line.

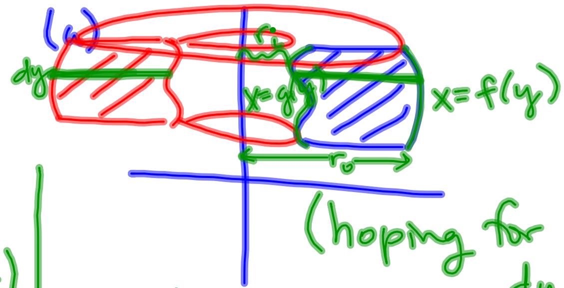
a) The *y*-axis

b) The *x*-axis

c) The line *y = 3*



(a)



(hoping for a dy)

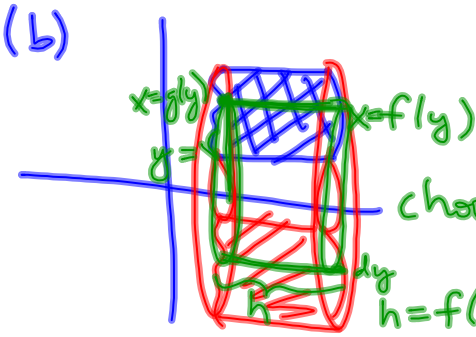
choose washer method

$$V = \pi \int_a^b (r_o^2 - r_i^2)\,dy$$

$$= \pi \int_c^d (r_o^2 - r_i^2)\,dy$$

$$\boxed{V = \pi \int_c^d \left((f(y))^2 - (g(y))^2 \right) dy}$$

(b)



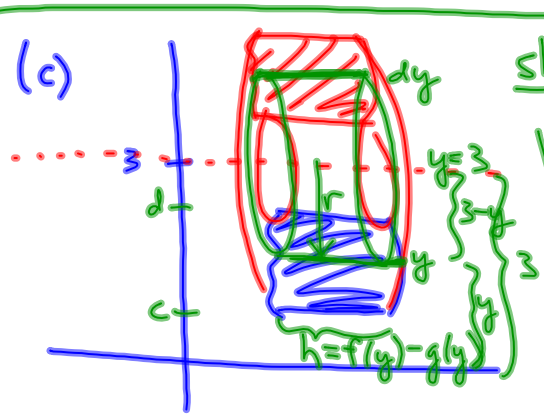
choose shell method

h=f(y)−g(y)

$$V = 2\pi \int_a^b rh\,dy = 2\pi \int_c^d rh\,dy$$

$$\boxed{V = 2\pi \int_c^d y(f(y) - g(y))\,dy}$$

(c)



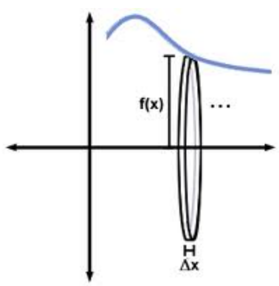
shell

$$V = 2\pi \int_a^b rh\,dy$$

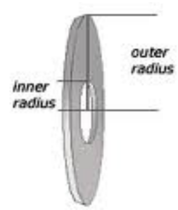
$$= 2\pi \int_c^d rh\,dy$$

$$\boxed{V = 2\pi \int_c^d (3-y)(f(y) - g(y))\,dy}$$

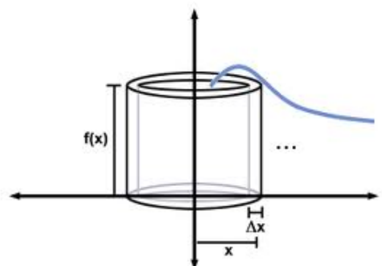
30B Volume Solids



disk



washer



shell