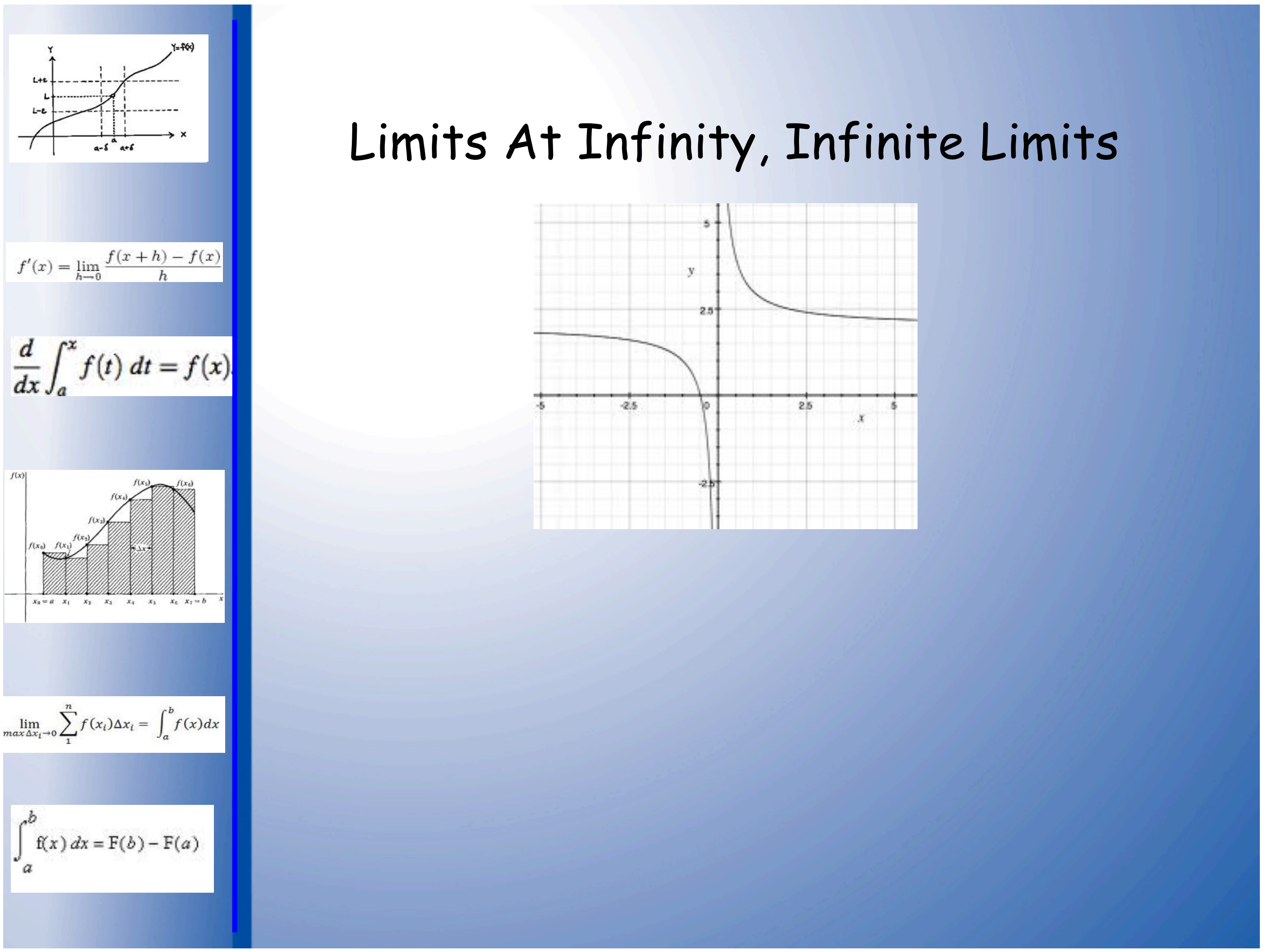

Limits At Infinity, Infinite Limits

4B Limits at Infinity

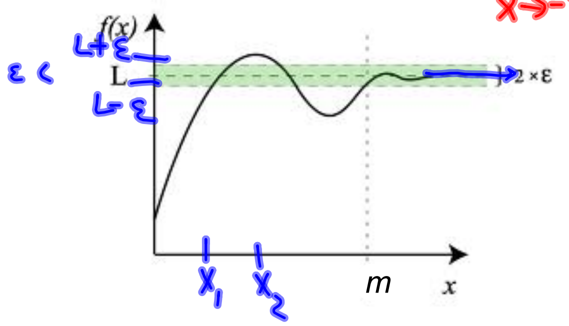
Definition: (Limit as $x \to \infty$)

$f(x)$ is defined on $[c, \infty)$ $\quad c \in \Re$ $\quad$ (or $(-\infty, c]$)

We say that if for every $\varepsilon > 0$ there is a corresponding number, m such that

$x > m \Rightarrow |f(x) - L| < \varepsilon,$ then $\lim_{x \to \infty} f(x) = L$

$(x < m)$ $\quad$ $(\lim_{x \to -\infty} f(x) = L)$



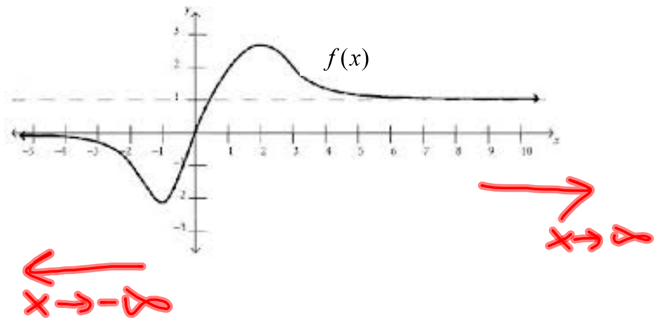
as x gets bigger the y-value ($f(x)$) gets closer & closer to L.

4B Limits at Infinity

EX 1 Intuitively (looking at the graph) determine these limits.

$\lim_{x \to \infty} f(x) = 1$

$\lim_{x \to -\infty} f(x) = 0$



EX 2 Show that if n is a positive integer, then $\lim_{x \to \infty} \frac{1}{x^n} = 0$.

Let $\varepsilon > 0$ be given. Choose $M = \left(\frac{1}{\varepsilon}\right)^{\frac{1}{n}} = \sqrt[n]{\frac{1}{\varepsilon}}$.

Then $x > M \Rightarrow \frac{x^n}{x^n M^n} > \frac{M^n}{x^n M^n}$

$\Leftrightarrow \frac{1}{M^n} > \frac{1}{x^n}$

So $\left|\frac{1}{x^n} - 0\right| = \left|\frac{1}{x^n}\right| < \frac{1}{M^n} = \frac{1}{\left(\sqrt[n]{\frac{1}{\varepsilon}}\right)^n} = \frac{1}{1/\varepsilon} = \varepsilon$

i.e. $\left|\frac{1}{x^n} - 0\right| < \varepsilon$.

$\Rightarrow$ by defn $\lim_{x \to \infty} \frac{1}{x^n} = 0$ $(n \in \mathbb{Z}^+)$ //

positive integers

element of

EX 3 $\lim_{x\to\infty} \frac{2x+3}{x^2+1} = \lim_{x\to\infty} \left(\frac{2x+3}{x^2+1}\right)\left(\frac{1/x^2}{1/x^2}\right)$

highest degree term is x^2

$= \lim_{x\to\infty} \frac{\cancel{2/x} + \cancel{3/x^2}}{1 + \cancel{1/x^2}} = \frac{0}{1} = 0$

$\lim_{x\to\infty} \frac{2x+3}{x^2+1} = \lim_{x\to\infty} \frac{2x}{x^2} = \lim_{x\to\infty} \frac{2}{x} = 0$

EX 4 $\lim_{x\to\infty} \frac{3x^4 - 2x^3 + 53}{x^3+7} = \lim_{x\to\infty} \frac{3x^4}{x^3}$

$= \lim_{x\to\infty} 3x \to \boxed{\infty}$

Note of warning:

This shortcut works only when we have limit as $x \to \pm\infty$!!!

EX 5 $\lim_{x\to\infty} \frac{2x^2+5x-1}{x^2+3x} = \lim_{x\to\infty} \frac{2x^2}{x^2}$

$= \lim_{x\to\infty} 2 = \boxed{2}$

Limits of Rational Fns as $x \to \pm\infty$

$\lim_{x\to\pm\infty} \frac{n(x)}{d(x)}$

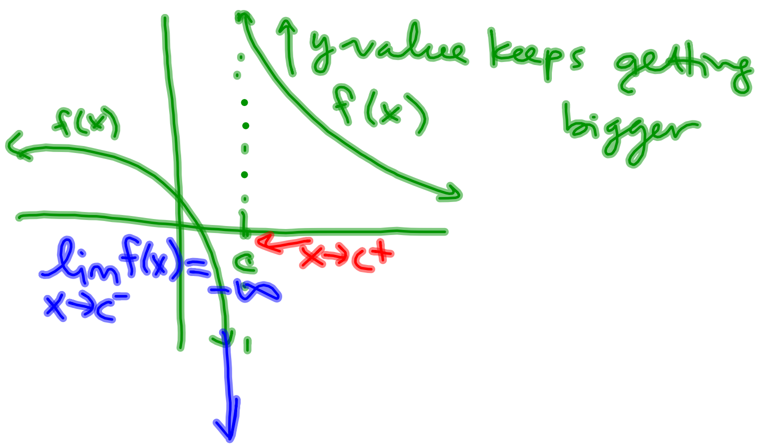
1. if degree of $n(x)$ < degree of $d(x)$, then limit goes to 0.
2. if degree of $n(x)$ > degree of $d(x)$, then limit goes to $\pm\infty$ (have to look at coefficients of leading terms)
3. if degree of $n(x)$ = degree of $d(x)$, then limit is quotient of the leading coefficients.

Definition: (Infinite limit)

We say $\lim\limits_{x \to c^+} f(x) = \infty$ if for every positive number, m when x is close to c

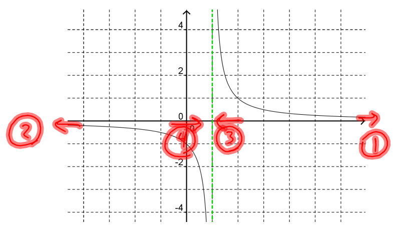
there is a corresponding $\delta > 0$ such that $0 < x - c < \delta \Rightarrow f(x) > m$ → f(x) keeps growing

($x \to c^+$: means x is going to c from the right)



4B Limits at Infinity

EX 6 Determine these limits looking at this graph of $f(x)=\dfrac{1}{x-1}$. (VA at $x=1$)



① $\lim\limits_{x\to\infty} f(x) = 0$ ③ $\lim\limits_{x\to 1^+} f(x) = \infty$

② $\lim\limits_{x\to -\infty} f(x) = 0$ ④ $\lim\limits_{x\to 1^-} f(x) = -\infty$

Ex 7 Find the horizontal and vertical asymptotes for this function, then write a few limit statements including ∞. $f(x)=\dfrac{-2x}{x+3}$

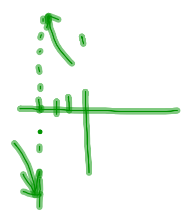
VA: of form $x=c$, c is a constant
(the graph of $y=f(x)$ can never cross or touch the VA)

find it by looking at domain restrictions

$f(x)$ has problem at $x=-3$ (make den. zero)

$\Rightarrow$ VA: $x=-3$ (domain: $x\in\mathbb{R}, x\neq -3$)

$\lim\limits_{x\to -3^+} \dfrac{-2x}{x+3} = \infty$ test: $x=-2.9$ $\dfrac{+}{+}$

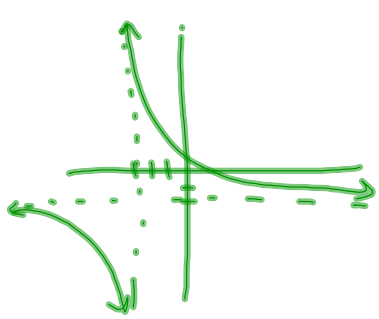


$\lim\limits_{x\to -3^-} \dfrac{-2x}{x+3} = -\infty$ test: $x=-3.1$ $\dfrac{+}{-}$

HA: horiz. line that fn approaches, eventually (as x gets huge)

$y = \lim\limits_{x\to\pm\infty} f(x)$

$y = \lim\limits_{x\to\pm\infty} \dfrac{-2x}{x+3} = \lim\limits_{x\to\pm\infty} \dfrac{-2x}{x} = -2 \Rightarrow$ HA: $y=-2$



Ex 8 a) Find the vertical and horizontal asymptotes for this function.

$$f(x) = \frac{2x}{\sqrt{x^2+5}}$$

VA: (domain restrictions) none

b) Determine these limits:

$\lim_{x\to\infty} f(x) =$

HA: $\lim_{x\to\infty} \frac{2x}{\sqrt{x^2+5}}$

$= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2}} = \lim_{x\to\infty} \frac{2x}{x}$

$= \lim_{x\to\infty} 2 = 2$

$\Rightarrow \boxed{\text{HA: } y=2}$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{2x}{\sqrt{x^2}}$

$= \lim_{x\to-\infty} \frac{2\cancel{x}}{-\cancel{x}}$

$= \lim_{x\to-\infty} -2$

$= -2$

$\Rightarrow \boxed{\text{HA: } y=-2}$

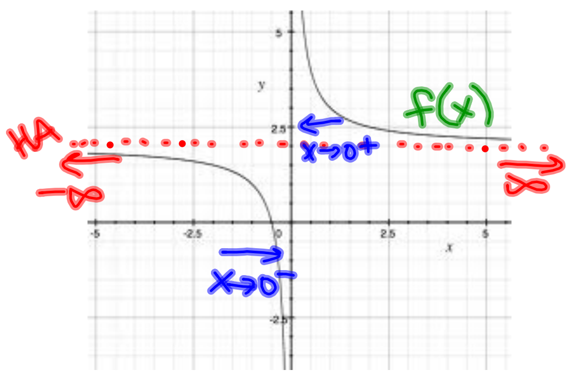
Note: $\sqrt{x^2} = |x| = \begin{cases} x, & \text{if } x \ge 0 \\ -x, & \text{if } x<0 \end{cases}$

ex $\sqrt{(-5)^2} = 5$

Note: HA(s) describe behavior of y-value as x gets huge!!!
(we can cross the HA as many times as fn requires when x is not "huge")

4B Limits at Infinity

Determine these limits:



$$\lim_{x\to\infty} f(x) = 2$$

$$\lim_{x\to 0^+} f(x) = \infty$$

$$\lim_{x\to -\infty} f(x) = 2$$

$$\lim_{x\to 0^-} f(x) = -\infty$$

$\lim_{x\to 0} f(x)$ DNE