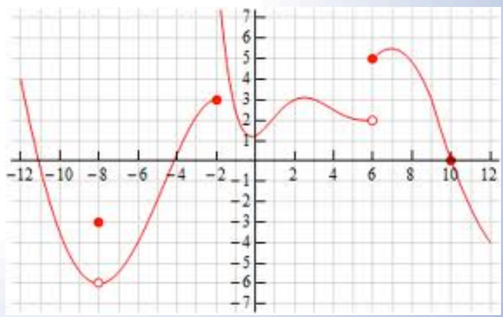
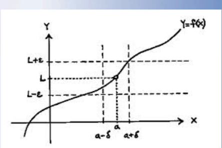
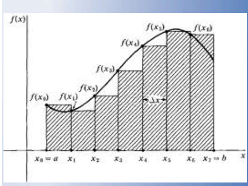
Continuity





$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$



$$\lim_{max\,\Delta x_i \to 0} \sum_1^n f(x_i)\Delta x_i = \int_a^b f(x)dx$$

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Definition: Continuity at a Point

(Continuity means we can draw graph of $f(x)=y$ w/ one swype)

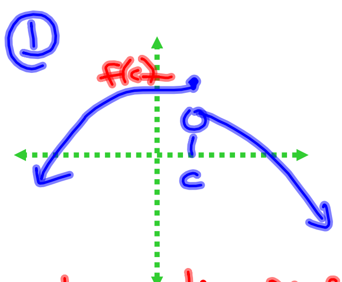
Let *f* be defined on an open interval containing *c*. We say that *f* is continuous at *c* if

$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

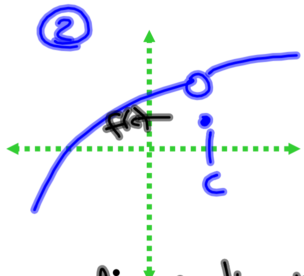
This indicates three things:

1. The function is defined at *x* = *c*. i.e. $f(c)$ exists
2. The limit exists at *x* = *c*. i.e. $\lim_{x \to c} f(x)$ exists
3. The limit at *x* = *c* needs to be exactly the value of the function at x = c. $\lim_{x \to c} f(x) = f(c)$.

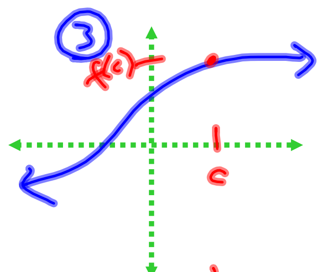
Three examples:



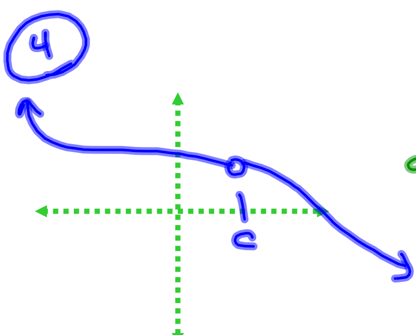
discontinuous at $x=c$.
✓ $f(c)$ exists but $\lim_{x \to c} f(x)$ DNE
fails condit'n (2)



discontinuity at $x=c$
$f(c)$ exists
$\lim_{x \to c} f(x)$ exists
$\lim_{x \to c} f(x) \neq f(c)$



continuous everywhere
$f(c)$ exists
$\lim_{x \to c} f(x)$ exists
✓ $\lim_{x \to c} f(x) = f(c)$



discontinuous at $x=c$
$\lim_{x \to c} f(x)$ exists
but $f(c)$ DNE

Continuous Functions

a) All polynomial functions are continuous everywhere.

b) All rational functions are continuous over their domain. ← (except x-values that make denominator = 0)

c) The absolute value function is continuous everywhere.

d) $f(x)=\sqrt[n]{x}$ is continuous for all real numbers if n is odd.

e) $f(x)=\sqrt[n]{x}$ is continuous for all non-negative real numbers if n is even.

f) The sine and cosine functions are continuous over all real numbers.

g) The cotangent, cosecant, secant and tangent functions are continuous over their domain.

More continuous functions

If *f(x)* and *g(x)* are continuous at *x* = *c*, then so are

$$\underbrace{kf(x),\ (f\pm g)(x),\ (fg)(x),\ \frac{f}{g}(x)}_{}, (g(x)\neq 0),$$

$$f^n(x),\ \sqrt[n]{f(x)},\ (f(c)>0\ if\ n\ is\ even).$$

arithmetic combinations of cont. fns are also cont.

EX 1 State where these functions are continuous.

a) $f(x)=x^2-9$ (polynomial)

continuous everywhere

(i) $x \in \mathbb{R}$ or (ii) $(-\infty,\infty)$

b) $g(x)=\sqrt{x-5}$

need $x-5 \geq 0 \Leftrightarrow$ (i) $x \geq 5$ or (ii) $[5,\infty)$

c) $h(x)=\dfrac{21-7x}{x-3}$

need $x-3 \neq 0 \Leftrightarrow x \neq 3$

(i) $x \in \mathbb{R}, x \neq 3$

(ii) $(-\infty,3) \cup (3,\infty)$

($\cup$ "or")

d) $p(x)=\begin{cases} 7-3x & x \leq 3 \\ -2 & x>3 \end{cases}$

(piecewise fn)

each piece is polynomial

$\Rightarrow$ continuous everywhere, except possibly where the pieces "meet".

$p(3)=7-3(3)=7-9=-2$

$\lim_{x\to 3^+} p(x)=-2$

(bottom piece)

$\Rightarrow$ fn cont. at $x=3$

$\Rightarrow$ cont. (i) $x \in \mathbb{R}$

(ii) $(-\infty,\infty)$

Common Domain Restrictions

1. cannot divide by zero
2. cannot take even root of negative #
3. cannot take log of a non-positive #
4. (possible problem w/ continuity) where pieces of piecewise fn meet

Composite Limit Theorem

If $\lim_{x \to c} g(x) = L$ and *f* is continuous at *L*, then

$$\lim_{x \to c} f\left(g(x)\right) = f\left(\lim_{x \to c} g(x)\right) = f(L)$$

exchange order of limit w/ f

Ex 2 At what points are the following functions continuous?

a) $h(x) = \dfrac{1}{\sqrt{4+x^2}}$

$4+x^2 \geq 0$ for all x

$4+x^2 \neq 0 \Rightarrow$ so there are no problems

$\Rightarrow$ cont. everywhere

(i) $x \in \mathbb{R}$ or (ii) $(-\infty, \infty)$

b) $g(t) = |t-2|$

continuous everywhere

(i) $t \in \mathbb{R}$ or (ii) $(-\infty, \infty)$

Ex 3 If $f(x)=\frac{x^2-49}{x-7}$, how do we need to complete the definition for this to be continuous everywhere?

note: right now, it has discont. pt at $x=7$. ($f(7)$ DNE)

$$\lim_{x\to 7}\frac{x^2-49}{x-7} = \lim_{x\to 7}\frac{\cancel{(x-7)}(x+7)}{\cancel{(x-7)}}$$

($\frac{0}{0}$ case)

$$= \lim_{x\to 7}(x+7) = 7+7=14$$

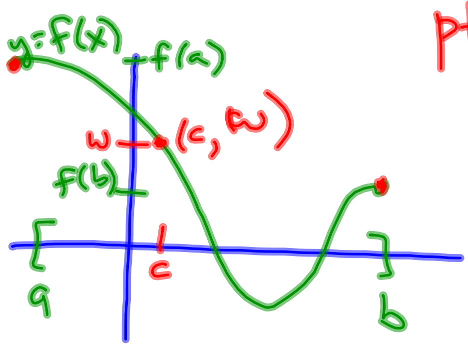
"patch hole" by making $f(7)=14$

$$f(x)=\begin{cases}\frac{x^2-49}{x-7}, & x\neq 7\\ 14, & x=7\end{cases}$$

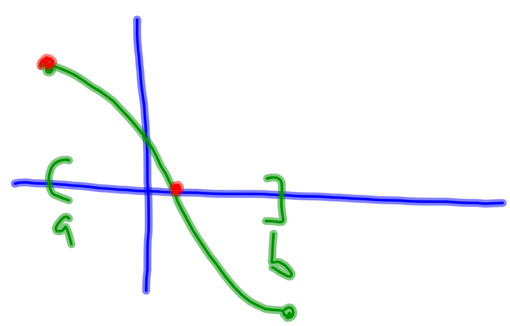
<u>Intermediate Value Theorem</u>

f is a function defined on *[a,b]* and *ω* is a number between *f(a)* and *f(b)*.

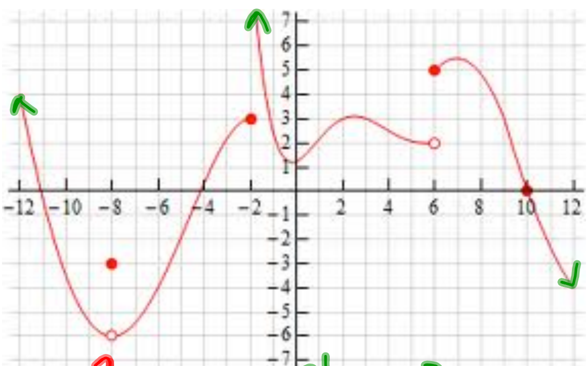
If *f* is continuous on *[a,b]*, then there exists at least one number, *c, (a<c<b)* such that *f(c) = ω*.



pt in red exists



Use interval notation to state all values for which this function is continuous.



cont. on

$(-\infty,-8)\cup(-8,-2)$

$\cup(-2,6)$

$\cup(6,\infty)$

discontinuous

at $x=-8,-2,6$

at $x=-8$

$f(-8)$ exists

$\lim_{x\to-8} f(x)$ exists

but $f(-8)\neq\lim_{x\to-8} f(x)$

at $x=-2$

$\lim_{x\to 2} f(x)$ DNE

$f(2)$ exists

at $x=6$

$\lim_{x\to 6} f(x)$ DNE

$f(6)$ exists