Further Practice on Techniques of Integration

$$\int \frac{f(x)}{g(x)}\,dx \qquad \int g(x)f(x)\,dx$$

If

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{0}{0}$$

or

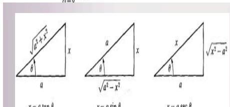
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

Then

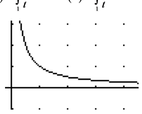
$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad + \frac{f'''(x_0)}{3!}(x-x_0)^3 + \frac{f''''(x_0)}{4!}(x-x_0)^4 + \cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x) = \int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2) = \int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv - \int v\,du$$

the product rule for differentiation

where it comes from: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

put into reverse $\int \frac{d}{dx}(uv) = \int (u\frac{dv}{dx} + v\frac{du}{dx})$

$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$

and then rearranged $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$

EX 1 $\displaystyle\int \frac{1}{\sqrt{x}+\sqrt[3]{x}}\,dx = \int \frac{1}{x^{1/2}+x^{1/3}}\,dx$

$u = x^{1/6} = \sqrt[6]{x}$ $\quad u^6 = x$

$du = \frac{1}{6}x^{-5/6}\,dx$ $\quad u^3 = x^{1/2}$

$\quad u^2 = x^{1/3}$

$6\,du = \frac{1}{\sqrt[6]{x^5}}\,dx$

$6\,du = \frac{1}{u^5}\,dx$

$6u^5\,du = dx$

$$= \int \frac{1}{u^3+u^2}(6u^5)\,du$$

$$= 6\int \frac{u^5}{u^2(u+1)}\,du$$

$$= 6\int \frac{u^3}{u+1}\,du$$

(improper rational fn)

$w = u+1 \iff u = w-1$

$dw = du$

$$= 6\int \frac{(w-1)^3}{w}\,dw$$

$$= 6\int \frac{w^3-3w^2+3w-1}{w}\,dw$$

$w = \sqrt[6]{x}+1$

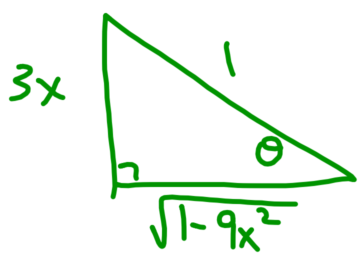
$$= 6\int \left(w^2-3w+3-\frac{1}{w}\right)dw$$

$$= 6\left(\frac{w^3}{3}-\frac{3w^2}{2}+3w-\ln|w|\right)+C$$

$$= 2\left(\sqrt[6]{x}+1\right)^3 - 9\left(\sqrt[6]{x}+1\right)^2 + 18\left(\sqrt[6]{x}+1\right) - 6\ln\left|\sqrt[6]{x}+1\right| + C$$

EX 2 $\displaystyle\int \frac{x^2}{\left(1-9x^2\right)^{3/2}}\,dx = \int \frac{x^2}{\left(\sqrt{1-9x^2}\right)^3}\,dx$

(sq. root of quadratic polynomial $\rightarrow$ trig sub.) try



$\frac{3x}{1} = \sin\theta$

$x = \frac{1}{3}\sin\theta$

$dx = \frac{1}{3}\cos\theta\, d\theta$

$\frac{\sqrt{1-9x^2}}{1} = \sqrt{1-9x^2} = \cos\theta$

$= \frac{1}{3}\displaystyle\int \frac{\left(\frac{1}{3}\sin\theta\right)^2}{(\cos\theta)^{\cancel{3}\,2}}\cancel{\cos\theta}\, d\theta$

$= \frac{1}{27}\displaystyle\int \frac{\sin^2\theta}{\cos^2\theta}\, d\theta$

$= \frac{1}{27}\displaystyle\int \tan^2\theta\, d\theta$

$= \frac{1}{27}\displaystyle\int \left(\sec^2\theta - 1\right) d\theta$

$= \frac{1}{27}\left(\tan\theta - \theta\right) + C = \boxed{\frac{1}{27}\left(\frac{3x}{\sqrt{1-9x^2}} - \arcsin(3x)\right) + C}$

EX 3 $\displaystyle\int \frac{2x^4 - 2x^3 + 6x^2 - 5x + 1}{x^3 - x^2 + x - 1}\,dx$ (improper rational fn)

$\Rightarrow$ need long division.

$$\begin{array}{r|l}
 & \qquad\qquad\qquad 2x \\
x^3 - x^2 + x - 1 & 2x^4 - 2x^3 + 6x^2 - 5x + 1 \\
 & -(2x^4 - 2x^3 + 2x^2 - 2x) \\
\hline
 & \qquad\qquad 4x^2 - 3x + 1
\end{array}$$

$$= \int \left(2x + \frac{4x^2 - 3x + 1}{x^3 - x^2 + x - 1}\right) dx$$

$$\begin{array}{r|rrrr}
1 & 1 & -1 & 1 & -1 \\
 & & 1 & 0 & 1 \\
\hline
 & 1 & 0 & 1 & 0
\end{array}$$

$\Rightarrow x^3 - x^2 + x - 1$

$= (x-1)(x^2+1)$

$$= \int \left(2x + \frac{4x^2 - 3x + 1}{(x-1)(x^2+1)}\right) dx$$

$$= x^2 + \int \frac{4x^2 - 3x + 1}{(x-1)(x^2+1)}\,dx$$

do PFD:

$$\frac{4x^2 - 3x + 1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx + C}{x^2+1}$$

$$4x^2 - 3x + 1 = A(x^2+1) + (Bx + C)(x-1)$$

$x = 1$: $4 - 3 + 1 = 2A$

$2 = 2A$

$A = 1$

$x = -1$: $4 + 3 + 1 = 2A + B(-1)(-2)$

$8 = 2 + 2B$

$6 = 2B$

$B = 3$

$x = 0$: $1 = A + C(-1)$

$1 = 1 - C$

$C = 0$

integral:

$$= x^2 + \int \left(\frac{1}{x-1} + \frac{3x}{x^2+1}\right) dx$$

$$= x^2 + \frac{\ln|x-1|}{1} + 3\int \frac{x}{x^2+1}\,dx$$

$u = x^2 + 1$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$$= x^2 + \ln|x-1| + \frac{3}{2}\int \frac{du}{u}$$

$$= x^2 + \ln|x-1| + \frac{3}{2}\ln|u| + C = \boxed{x^2 + \ln|x-1| + \frac{3}{2}\ln(x^2+1) + C}$$

EX 4 $\int \frac{e^{2x}}{\sqrt[3]{1+e^x}}\,dx = \int \frac{e^x e^x\,dx}{\sqrt[3]{1+e^x}}$

$u = 1+e^x$

$du = e^x dx$

$\Rightarrow e^x = u-1$

$= \int \frac{(u-1)\,du}{\sqrt[3]{u}}$

$= \int \left(\frac{u}{\sqrt[3]{u}} - \frac{1}{\sqrt[3]{u}}\right) du$

$= \int \left(u^{2/3} - u^{-1/3}\right) du$

$= \frac{3}{5}u^{5/3} - \frac{3}{2}u^{2/3} + C$

$= \boxed{\frac{3}{5}(1+e^x)^{5/3} - \frac{3}{2}(1+e^x)^{2/3} + C}$

EX 5 $\int \cos(\sqrt{x})\,dx$ $= \int \cos u\,(2u\,du)$

$u = \sqrt{x}$

$du = \frac{1}{2\sqrt{x}}\,dx$

$du = \frac{1}{2u}\,dx$

$2u\,du = dx$

$= 2\int u\cos u\,du$ (classic integration by parts)

$w = u$ $v = \sin u$

$dw = du$ $dv = \cos u\,du$

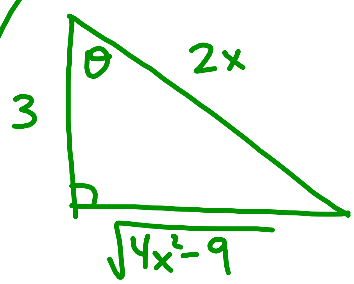
$= 2\left(u\sin u - \int \sin u\,du\right)$

$= 2u\sin u + 2\cos u + C$

$= \boxed{2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C}$

EX 6 $\int \frac{\sqrt{4x^2-9}}{x}\,dx$

(classic trig sub. sq. root of quadratic)



$\frac{2x}{3} = \sec\theta$

$x = \frac{3}{2}\sec\theta$

$dx = \frac{3}{2}\sec\theta\tan\theta\,d\theta$

$\left(\frac{3}{2x} = \cos\theta \quad \theta = \cos^{-1}\left(\frac{3}{2x}\right)\right)$

$\frac{\sqrt{4x^2-9}}{3} = \tan\theta \Rightarrow \sqrt{4x^2-9} = 3\tan\theta$

$= \int \frac{3\tan\theta}{\cancel{\frac{3}{2}}\cancel{\sec\theta}}\left(\cancel{\tfrac{3}{2}}\cancel{\sec\theta}\tan\theta\right)d\theta$

$= 3\int \tan^2\theta\,d\theta$

$= 3\int (\sec^2\theta - 1)\,d\theta = 3(\tan\theta - \theta) + C$

$= 3\tan\theta - 3\theta + C$

$= \boxed{\sqrt{4x^2-9} - 3\cos^{-1}\left(\frac{3}{2x}\right) + C}$

EX 7 $\displaystyle\int \frac{\sqrt[3]{x+8}}{x}\,dx$

(we have root of linear polynomial)

try $u = \sqrt[3]{x+8} = (x+8)^{1/3} \Rightarrow u^3 = x+8$

$x = u^3 - 8$

$$du = \frac{1}{3(x+8)^{2/3}}\,dx$$

$$3\,du = \frac{1}{\left((x+8)^{1/3}\right)^2}\,dx$$

$$3\,du = \frac{1}{u^2}\,dx$$

$$3u^2\,du = dx$$

$$= \int \frac{u}{u^3-8}\left(3u^2\,du\right)$$

$$= 3\int \frac{u^3}{u^3-8}\,du$$

(improper rational fn $\Rightarrow$ need long division)

$$= 3\int \frac{u^3-8+8}{u^3-8}\,du$$

$$= 3\int\left(\frac{u^3-8}{u^3-8} + \frac{8}{u^3-8}\right)du = 3\int\left(1 + \frac{8}{u^3-8}\right)du$$

need PFD

$$\frac{8}{u^3-8} = \frac{8}{(u-2)(u^2+2u+4)} = \frac{A}{u-2} + \frac{Bu+C}{u^2+2u+4}$$

$$8 = A(u^2+2u+4) + (Bu+C)(u-2)$$

$u=2$: $8 = A(4+4+4)$

$8 = 12A$

$A = \frac{2}{3}$

$u=1$: $8 = \frac{2}{3}(7) + (B - \frac{8}{3})(-1)$

$8 = \frac{14}{3} - B + \frac{8}{3}$

$\frac{2}{3} = -B$

$B = -2/3$

$u=0$: $8 = 4A + C(-2)$

$8 = \frac{8}{3} - 2C$

$\frac{16}{3} = -2C$

$-\frac{8}{3} = C$

integral:

$$= 3\int\left(1 + \frac{2/3}{u-2} + \frac{-2/3\,u - 8/3}{u^2+2u+4}\right)du$$

$$= \int\left(3 + \frac{2}{u-2} - \frac{2u+8}{u^2+2u+4}\right)du$$

$$(*) = 3u + 2\ln|u-2| - \int \frac{2u+8}{u^2+2u+4}\,du$$

$w = u^2+2u+4$

$dw = (2u+2)\,du$

$$\int \frac{2u+8}{u^2+2u+4}\,du = \int \frac{2u+2+6}{u^2+2u+4}\,du = \int \frac{2u+2}{u^2+2u+4}\,du + \int \frac{6}{u^2+2u+4}\,du$$

$$= \int \frac{dw}{w} + 6\int \frac{1}{(u+1)^2 + (\sqrt{3})^2}\,du$$

u^2+2u+4
$= (u^2+2u+1) + 3$
$= (u+1)^2 + 3$

$$= \ln|w| + 6\left(\frac{1}{\sqrt{3}}\right)\arctan\left(\frac{u+1}{\sqrt{3}}\right) + C$$

$$= \ln|u^2+2u+4| + \frac{6}{\sqrt{3}}\arctan\left(\frac{u+1}{\sqrt{3}}\right) + C$$

$$(*) \Rightarrow 3u + 2\ln|u-2| - \ln|u^2+2u+4| - \frac{6}{\sqrt{3}}\arctan\left(\frac{u+1}{\sqrt{3}}\right) + C$$

$$= 3\sqrt[3]{x+8} + 2\ln\left|\sqrt[3]{x+8} - 2\right| - \ln\left|\left(\sqrt[3]{x+8}\right)^2 + 2\sqrt[3]{x+8} + 4\right| - \frac{6}{\sqrt{3}}\arctan\left(\frac{\sqrt[3]{x+8}+1}{\sqrt{3}}\right) + C$$