If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

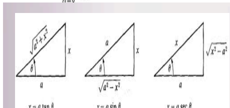
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

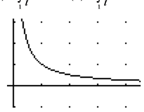
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$f(x) = f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots = \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Sequences and Series Review

2,4,6,8,10,...

$$a_n = a_1 + (n-1)d$$

$$\sum_{k=1}^{6} a_1 + (k-1)d$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$

2,4,8,16,32,...

$$a_n = a_1 r^{(n-1)}$$

$$\sum_{k=1}^{6} a_1 r^{k-1}$$

$$S_n = a_1\frac{1-r^n}{1-r}$$

A ***sequence*** $\{a_n\}$ is a function such that the domain is the set of positive integers and the range is a set of real numbers.

Write five terms for each of these sequences:

$$a_n = \frac{n}{2n+1}$$

$a_1 = \frac{1}{3}$, $a_2 = \frac{2}{5}$, $a_3 = \frac{3}{7}$, $a_4 = \frac{4}{9}$, $a_5 = \frac{5}{11}$

$$a_n = \frac{(-2)^n}{n!}$$

$-2, 2, \frac{-4}{3}, \frac{2}{3}, \frac{-4}{15}, \ldots$

$a_1 = \frac{-2}{1!} = -2$

$a_2 = \frac{(-2)^2}{2!} = \frac{4}{2} = 2$

$a_3 = \frac{(-2)^3}{3!} = \frac{-8}{3 \cdot 2 \cdot 1} = \frac{-4}{3}$

$a_4 = \frac{(-2)^4}{4!} = \frac{16}{8 \cdot 3} = \frac{2}{3}$

$a_5 = \frac{(-2)^5}{5!} = \frac{-32}{5 \cdot 4 \cdot 3 \cdot 2} = \frac{-4}{15}$

A ***series*** is the sum of a sequence. $\sum_{k=1}^{n} a_k$

A ***partial sum*** is the sum of the first n terms.
An ***infinite sum*** is the sum from k= 1 to ∞.

Find these partial sums:

$$\sum_{k=0}^{3} \frac{(-2)^k}{k!}$$

$= \frac{(-2)^0}{0!} + \frac{(-2)^1}{1!} + \frac{(-2)^2}{2!} + \frac{(-2)^3}{3!}$

(k=0) (k=1) (k=2) (k=3)

$= 1 + -2 + \frac{4}{2} + \frac{-8}{6}$

$= 1 - 2 + 2 - \frac{4}{3} = \frac{-1}{3}$

$$\sum_{k=1}^{5} \frac{k}{2k+1}$$

$= \left(\frac{1}{2+1}\right) + \left(\frac{2}{4+1}\right) + \left(\frac{3}{6+1}\right) + \left(\frac{4}{8+1}\right) + \left(\frac{5}{10+1}\right)$

$= \frac{1}{3} + \frac{2}{5} + \frac{3}{7} + \frac{4}{9} + \frac{5}{11}$

$= \frac{7141}{3465}$

Arithmetic Sequence, Series

d = common difference

$$a_n = a_1 + (n-1)d$$ nth term of arithmetic sequence

ex 2, 5, 8, 11, 14, . . .

$d = 3$

Geometric Sequence, Series

r = common ratio

$$a_n = a_1 r^{(n-1)}$$ nth term in geometric sequence

ex 2, 6, 18, 54, ...

$r = 3$

partial sum

$$\sum_{k=1}^{n} (a_1 + (k-1)d)$$

$$S_n = \frac{n}{2}(a_1 + a_n)$$ nth partial sum

arithmetic

$$\sum_{k=1}^{n} a_1 r^{k-1}$$

$$S_n = a_1 \frac{1-r^n}{1-r}$$

geometric

infinite sum

$$\sum_{k=1}^{\infty} a_1 r^{k-1} = \sum_{k=0}^{\infty} a_1 r^k$$

$$S_\infty = \frac{a_1}{1-r}$$ if $|r| < 1$

geometric

Determine the sum for each of these:

① $\sum_{k=1}^{50} (2k-3)$

(arithmetic series)

$S_{50} = \frac{50}{2}(a_1 + a_{50})$

$a_k = 2k - 3$

$a_1 = 2(1) - 3 = -1$

$a_{50} = 2(50) - 3 = 97$

$\Rightarrow S_{50} = 25(-1 + 97)$

$= 25(96)$

$= 2400$

② $\sum_{k=1}^{\infty} \left(\frac{2}{3}\right)^k$ $\quad r = \frac{2}{3} < 1$

(geometric)

$S_\infty = \frac{(2/3)}{1 - 2/3} = \frac{2/3}{1/3} = 2$

ex $\sum_{n=5}^{\infty} 3\left(\frac{1}{4}\right)^n$ $r = \frac{1}{4} < 1$

$$= \frac{3\left(\frac{1}{4}\right)^5}{1-\frac{1}{4}} = \frac{\frac{3}{4^5}}{\frac{3}{4}} = \frac{3}{4^5} \cdot \frac{4}{3} = \frac{1}{4^4}$$

ex $\sum_{m=9}^{\infty} -5\left(\frac{7}{8}\right)^{m+3}$ $r = 7/8 < 1$

$$= \frac{-5\left(\frac{7}{8}\right)^{9+3}}{1-7/8} = \frac{-5\left(\frac{7}{8}\right)^{12}}{\frac{1}{8}} = -5\left(\frac{7}{8}\right)^{12} \cdot \frac{8}{1}$$

$$= \frac{-5(7^{12})}{8^{11}}$$

ex $\sum_{k=4}^{\infty} 72\left(\frac{1}{3^{2k}}\right) = \sum_{k=4}^{\infty} 72\left(\frac{1}{3^2}\right)^k$

$$= \sum_{k=4}^{\infty} 72\left(\frac{1}{9}\right)^k$$ $r = 1/9 < 1$

$$= \frac{72\left(\frac{1}{9}\right)^4}{1-\frac{1}{9}} = 72\left(\frac{1}{9^4}\right) \cdot \frac{9}{8}$$

$$= 9\left(\frac{1}{9^3}\right) = \frac{1}{9^2} = \frac{1}{81}$$

Common Elements of Sequences/Series:

Odd numbers $1, 3, 5, 7, 9, \ldots$

$a_n = 2n+1$, $n = 0, 1, 2, \ldots$ or $a_n = 2n-1$, $n = 1, 2, \ldots$

Even numbers $2, 4, 6, 8, 10, \ldots$

$a_n = 2n$, $n = 1, 2, 3, \ldots$ or $a_n = 2n-2$, $n = 2, 3, 4, \ldots$

Factorials $1, 1, 2, 6, 24, 120, 720, \ldots$

$= 0!, 1!, 2!, 3!, 4!, 5!, 6!, \ldots$ $a_n = n!$, $n = 0, 1, \ldots$

or $a_n = (n-1)!$, $n = 1, 2, \ldots$

Alternating signs $1, -1, 1, -1, 1, -1, \ldots$

$a_n = (-1)^n$, $n = 0, 1, 2, \ldots$ or $a_n = (-1)^{n-1}$, $n = 1, 2, 3, \ldots$

Powers of 2 $1, 2, 4, 8, 16, 32, 64, \ldots$

$a_n = 2^n$, $n = 0, 1, 2, \ldots$ or $a_n = 2^{n-1}$, $n = 1, 2, 3, \ldots$

Arithmetic, Geometric or Neither?

n[th] term 20[th] term $a_n \to 0?$ $\sum_{k=1}^{\infty} a_k \to$ some value?

a) 1,1,2,3,5,8,13,..... (Fibonacci sequence)

$n=1$, $n=2$

$a_1 = 1,\ a_2 = 1,$

$a_n = a_{n-1} + a_{n-2}$ (recursive)

$a_{20} = a_{19} + a_{18}$

$a_n \not\to 0$ (a_n does not go to 0 as $n \to \infty$)

$\Rightarrow \sum_{k=1}^{\infty} a_k = \infty$

b) $a_1 = 2,\ a_{n+1} = \frac{a_n}{n}$, $n = 1, 2, \ldots$

$a_2 = \frac{a_1}{1} = \frac{2}{1} = 2$

$a_3 = \frac{a_2}{2} = \frac{2}{2} = 1$

$a_4 = \frac{a_3}{3} = \frac{1}{3}$

$a_5 = \frac{a_4}{4} = \frac{1}{12}$

$a_6 = \frac{a_5}{5} = \frac{1}{60}$

$2, 2, 1, \frac{1}{3}, \frac{1}{12}, \frac{1}{60}, \ldots$ (neither)

$a_n \to 0$ (as $n \to \infty$)

$\sum_{k=1}^{\infty} a_n$ finite? don't know

c) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$

$n=1$ $n=2$ $n=3$

$a_n = \frac{1}{2^n}$, $n = 1, 2, 3, \ldots$ $a_{20} = \frac{1}{2^{20}}$

$a_n \to 0$ as $n \to \infty$

$\sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1/2}{1 - 1/2} = 1$

(geometric series)

d) .9, .09, .009, .0009, ….

$n=1$ $n=2$ $n=3$

$= \frac{9}{10}, \frac{9}{100}, \frac{9}{1000}, \frac{9}{10000}, \ldots$

$= \frac{9}{10}, \frac{9}{10^2}, \frac{9}{10^3}, \frac{9}{10^4}, \ldots$

$a_n = \frac{9}{10^n} = 9\left(\frac{1}{10}\right)^n \quad \Rightarrow \quad \sum_{n=1}^{\infty} 9\left(\frac{1}{10}\right)^n = \frac{9(1/10)}{1 - 1/10} = \frac{9/10}{9/10} = 1$

geometric

$0.9 + 0.09 + 0.009 + \ldots = 1$

$\Leftrightarrow 0.\overline{9} = 1$

Write a formula for the n^{th} term of these sequences.

a) $\frac{1}{2}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \ldots$

n=1, n=2, n=3

$$a_n = \frac{2n-1}{2n}, \quad n = 1, 2, 3, \ldots$$

b) $\frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \ldots$

n=1, n=2

$$a_n = \frac{1}{2^{2^{n-1}}},$$

$n = 1, 2, 3, \ldots$

denominator:

$2, 2^2, 2^4, 2^8, 2^{16}, \ldots$

n	exponent
1	$1 = 2^0 = 2^{1-1}$
2	$2 = 2^1 = 2^{2-1}$
3	$4 = 2^2 = 2^{3-1}$
4	$8 = 2^3 = 2^{4-1}$
5	$16 = 2^4 = 2^{5-1}$

c) $\frac{-2}{1}, \frac{8}{2}, \frac{-26}{6}, \frac{80}{24}, \frac{-242}{120}$

n=1, n=2, n=3

(alternating signs)

$$a_n = \frac{(-1)^n (3^n - 1)}{n!},$$

$n = 1, 2, 3, \ldots$

or

$$a_n = \frac{(-1)^{n+1}(3^{n+1} - 1)}{(n+1)!}$$

$n = 0, 1, 2, \ldots$

numerator

$2, 8, 26, 80, 242, \ldots$

$= (3-1), (9-1), (27-1), (81-1), (243-1), \ldots$

$= (3^1-1), (3^2-1), (3^3-1), (3^4-1), (3^5-1), \ldots$

n=1, n=2, n=3

denominator

$1, 2, 6, 24, 120, \ldots$

$= 1!, 2!, 3!, 4!, 5!, \ldots$