If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

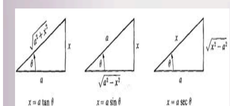
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

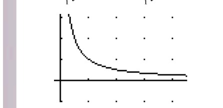
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv - \int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

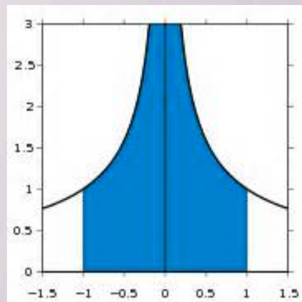
$$\int \frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

Improper Integrals With Indefinite Integrands



Improper Integrals: Infinite Integrands

Look at $\int_{-1}^{2} \frac{1}{x^4}\,dx$. Can we just do the integral?

there is a vertical asymptote for $y = \frac{1}{x^4}$ at $x=0$.

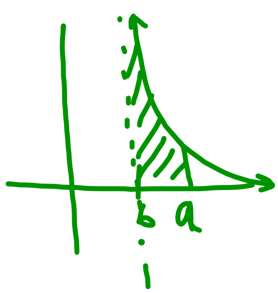
Definition

Let f(x) be continuous on [a,b) and

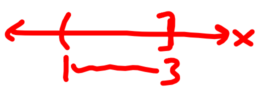
$$\lim_{x \to b^-} |f(x)| = \infty \Rightarrow \int_a^b f(x)dx = \lim_{t \to b^-} \int_a^t f(x)dx$$

(there is a vertical asymptote at $x=b$)

if the limit exists and is finite, otherwise it diverges.



EX 1 $\int_1^3 \frac{dx}{(x-1)^{4/3}} = \lim_{b\to 1^+} \int_b^3 \frac{dx}{(x-1)^{4/3}}$



$$= \lim_{b\to 1^+} -3(x-1)^{-1/3}\Big|_b^3$$

$$= \lim_{b\to 1^+} \frac{-3}{\sqrt[3]{x-1}}\Big|_b^3$$

$$= \frac{-3}{\sqrt[3]{3-1}} - \underbrace{\lim_{b\to 1^+} \frac{-3}{\sqrt[3]{b-1}}}_{\infty}$$

diverges

EX 2 $\int_0^9 \frac{dx}{\sqrt{9-x}} = \lim_{b\to 9^-} \int_0^b \frac{dx}{\sqrt{9-x}}$

$y = \frac{1}{\sqrt{9-x}}$ has VA at $x = 9$

$= \lim_{b\to 9^-} \int_0^b (9-x)^{-1/2}\,dx$

$= \lim_{b\to 9^-} \frac{2(9-x)^{1/2}}{-1}\Big|_0^b$

(if confused, do u-sub w/ $u = 9-x$)

$= \lim_{b\to 9^-} -2\sqrt{9-x}\,\Big|_0^b = \lim_{b\to 9^-} -2\cancel{\sqrt{9-b}} - -2\sqrt{9-0}$

$\cancel{\sqrt{9-b}} \to 0$

$= 2\sqrt{9} = \boxed{6}$

EX 3 $\int_0^1 \frac{1}{x^p}\,dx, \quad p \geq 1$

$y = \frac{1}{x^p}$, $p \geq 1$

there is VA at $x = 0$

$= \lim_{b\to 0^+} \int_b^1 \frac{1}{x^p}\,dx$

case 1: $p = 1$

$\lim_{b\to 0^+} \int_b^1 \frac{1}{x}\,dx = \lim_{b\to 0^+} \ln|x|\,\Big|_b^1$

$= \lim_{b\to 0^+} (\ln 1 - \cancel{\ln b})$

$\cancel{\ln b} \to \infty$

diverges

case 2: $p > 1$

$\lim_{b\to 0^+} \int_b^1 \frac{1}{x^p}\,dx = \lim_{b\to 0^+} \left(\frac{x^{-p+1}}{-p+1}\right)\Big|_b^1$

$= \frac{1^{-p+1}}{-p+1} - \lim_{b\to 0^+} \frac{b^{-p+1}}{-p+1}$

$= \frac{1}{-p+1} - \lim_{b\to 0^+} \frac{1}{-p+1}\left(\cancel{\frac{1}{b^{p-1}}}\right)$ diverges

$\cancel{\frac{1}{b^{p-1}}} \to \infty$

note:
$p > 1$
$p-1 > 0$
$-p+1 = -(p-1)$

Note: (belongs on your reference sheet)

① $\int_1^\infty \frac{1}{x^p}\,dx$ $\begin{cases} \text{converges if } p > 1 \\ \text{diverges if } p \leq 1 \end{cases}$

② $\int_0^1 \frac{1}{x^p}\,dx$ $\begin{cases} \text{diverges if } p \geq 1 \\ \text{converges if } p < 1 \end{cases}$

(note: if $0 < p < 1$, there's still a VA at $x = 0$.
if $p < 0$, then there is no VA, so the integral is finite.)

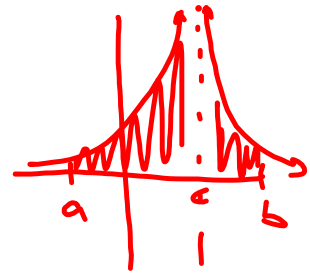
Definition

If f is continuous on $[a,b]$ except at $x=c$ where $a<b<c$

and $\lim_{x \to c} |f(x)| = \infty$ (there's a VA at $x=c$)

then $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

if both integrals converge. Otherwise it diverges.



EX 4 $\int_{-5}^{0} \frac{1}{(x+3)^2} dx$ $= \int_{-5}^{-3} \frac{1}{(x+3)^2} dx + \int_{-3}^{0} \frac{1}{(x+3)^2} dx$

$y = \frac{1}{(x+3)^2}$

there is a VA at $x=-3$

$= \lim_{b \to -3^-} \int_{-5}^{b} \frac{1}{(x+3)^2} dx + \lim_{a \to -3^+} \int_{a}^{0} \frac{1}{(x+3)^2} dx$

$= \lim_{b \to -3^-} \left(\frac{(x+3)^{-1}}{-1} \Big|_{-5}^{b} \right) + \lim_{a \to -3^+} \left(\frac{(x+3)^{-1}}{-1} \Big|_{a}^{0} \right)$

$= \lim_{b \to -3^-} \left(\frac{-1}{x+3} \Big|_{-5}^{b} \right) + \lim_{a \to -3^+} \left(\frac{-1}{x+3} \Big|_{a}^{0} \right)$

$= \lim_{b \to -3^-} \underbrace{\cancel{\frac{-1}{b+3}}}_{\infty} - \frac{-1}{-5+3} + \lim_{a \to -3^+} \left(\frac{-1}{0+3} - \underbrace{\cancel{\frac{-1}{a+3}}}_{\infty} \right)$

diverges

EX 5 $\int_{-3}^{1} \frac{5}{(x+2)^{3/5}}\,dx$ $= \lim_{a\to -2^-} \int_{-3}^{a} \frac{5}{(x+2)^{3/5}}\,dx + \lim_{b\to -2^+} \int_{b}^{1} \frac{5}{(x+2)^{3/5}}\,dx$

$$\boxed{\begin{array}{l} y=\frac{5}{(x+2)^{3/5}} \\ VA:\ x=-2 \end{array}}$$

$$= \lim_{a\to -2^-} \left(\frac{5(x+2)^{2/5}}{2/5} \Bigg|_{-3}^{a} \right)$$

$$+ \lim_{b\to -2^+} \left(\frac{5(x+2)^{2/5}}{2/5} \Bigg|_{b}^{1} \right)$$

$$= \lim_{a\to -2^-} \frac{25}{2}(\underbrace{a+2}_{0})^{2/5} - \frac{25}{2}(-3+2)^{2/5}$$

$$+ \frac{25}{2}(1+2)^{2/5} - \lim_{b\to -2^+} \frac{25}{2}(\underbrace{b+2}_{0})^{2/5}$$

$$= -\frac{25}{2}(1) + \frac{25}{2}(3^{2/5})$$

$$= \boxed{-\frac{25}{2}\left(1-3^{2/5}\right)}$$

Conclusion

- always be aware to check for VA in any definite integral