Alternating Series, Absolute Convergence and Conditional Convergence

If

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

or

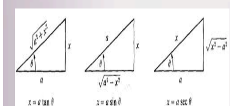
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

Then

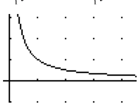
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &\quad + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f''''(x_0)}{4!}(x - x_0)^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \end{aligned}$$



$$\ln(x) = \int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2) = \int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv - \int v\,du$$

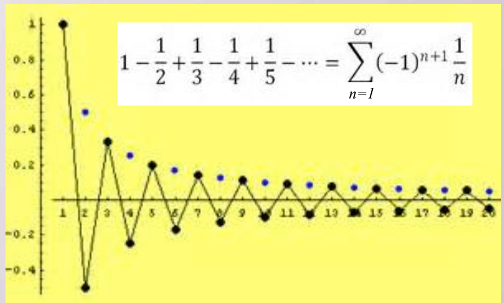
the product rule for differentiation

where it comes from: $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

put into reverse $\int \frac{d}{dx}(uv) = \int (u\frac{dv}{dx} + v\frac{du}{dx})$

$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$

and then rearranged $\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$



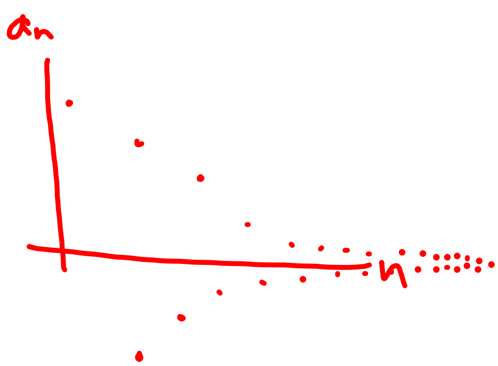
In an Alternating Series, every other term has the opposite sign.

$$a_1 - a_2 + a_3 - a_4 + a_5 - \ldots \qquad (a_i > 0)$$

AST (Alternating Series Test)

Let $a_1 - a_2 + a_3 - a_4 + \ldots$ be an alternating series such that $a_n > a_{n+1} > 0$, then the series converges. ($\Leftarrow$ if $\lim_{n\to\infty} a_n = 0$, then convergence)

The error made by estimating the sum, S_n is less than or equal to a_{n+1}, i.e. $E = |S - S_n| \le a_{n+1}$.



EX 1 Does an Alternating Harmonic Series converge or diverge?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

(we already know that harmonic series diverges, i.e. $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges)

AST: $\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{n} \right|$

$= \lim_{n\to\infty} \frac{1}{n} = 0 \Rightarrow$ by AST, this series converges

EX 2 Does this series diverge or converge? What is the error estimate made when approximating the sum using S_6?

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2+1}$$

(this is alternating series)

use AST: $\lim_{n\to\infty} \frac{n}{n^2+1} = 0 \Rightarrow$ by AST, this series converges.

$E_6 = |S - S_6| \le a_7 = \frac{7}{7^2+1} = \frac{7}{50} = \frac{14}{100} = 0.14$

Absolute Convergence

absolute conv.

$\Rightarrow$ cond. conv.

If $\sum |u_n|$ converges, then $\sum u_n$ converges.

EX 3 Does $\underset{n=1}{2} + \underset{n=2}{\frac{2}{2^3}} + \underset{n=3}{\frac{2}{3^3}} - \frac{2}{4^3} + \frac{2}{5^3} + \frac{2}{6^3} + \frac{2}{7^3} - \frac{2}{8^3} + \ldots$ converge or diverge?

(not alternating series)

$$\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{2}{n^3}$$

p-series $p=3>1$

$\Rightarrow$ convergence

(absolute convergence)

Absolute Ratio Test

Let $\sum a_n$ be a series of nonzero terms and suppose $\lim_{n\to\infty}\frac{|u_{n+1}|}{|u_n|}=\rho$.

i) if $\rho < 1$, the series converges absolutely.

ii) if $\rho > 1$, the series diverges.

iii) if $\rho = 1$, then the test is inconclusive.

EX 4 Show $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{e^n}$ converges absolutely.

(alternating series)

ART:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(n+1)^2}{e^{n+1}}\cdot\frac{e^n}{(-1)^{n+1}n^2}\right|$$

$$=\lim_{n\to\infty}\frac{(n+1)^2}{e^1 n^2}=\frac{1}{e}\lim_{n\to\infty}\frac{(n+1)^2}{n^2}$$

$$=\frac{1}{e}(1)=\frac{1}{e}<1 \Rightarrow \boxed{\text{series converges absolutely.}}$$

Conditional Convergence

$\sum u_n$ is conditionally convergent if $\sum u_n$ converges but $\sum |u_n|$ does not.

Note: if it's an all-positive series, then convergence means absolute convergence.

EX 5 Classify as absolutely convergent, conditionally convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$$

check for absolute convergence first:

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$$ use LCT, choose $b_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$

$\sum b_n$ p-series, $p = 1/2 < 1$

$\Rightarrow \sum b_n$ diverges

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} \cdot \frac{\sqrt{n}}{1}$$

$$= \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{2\sqrt{n}}$$

$$= \frac{1}{2} < \infty$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}}$ diverges $\Rightarrow$ this series is <u>not</u> absolutely convergent

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}+\sqrt{n}}$ still need to check for cond. convergence

use AST: $\lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

$\Rightarrow$ by AST, this series converges <u>conditionally</u>

Rearrangement Theorem

The terms of an absolutely convergent series can be rearranged without affecting either the convergence or the sum of the series.

(The terms of a conditionally convergent series or a divergent series **cannot** be rearranged w/out affecting the sum.)