If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

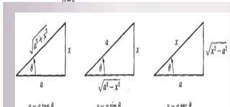
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

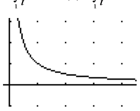
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

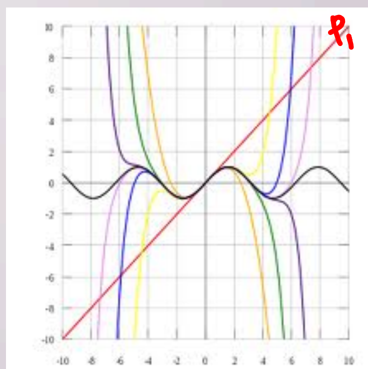
$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

The Taylor Approximation to a Function



black curve

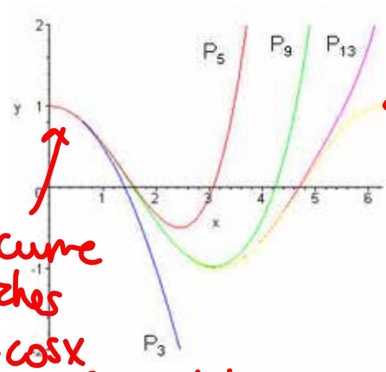
$y = \sin x$

Taylor Approximations to a Function

Many math problems that occur in applications cannot be solved exactly, like $\int_0^b \sin\left(x^2\right)dx$. We need to approximate them.

Taylor Polynomial of order n (based at a)

$$P_n(x) = f(a) + f'(a)\ (x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$



P_3 curve matches $y = \cos x$ curve for a bit

ex $f(x) = \cos x$ (in yellow)

$P_3 = 3^{rd}$ order Taylor polynomial for $f(x) = \cos x$ centered at $a = 0$.

EX 1 For $f(x) = e^{-3x}$, find the Maclaurin polynomial of order 4 and approximate $f(0.12)$.

We already know $e^{\heartsuit} = \sum_{n=0}^{\infty} \frac{\heartsuit^n}{n!}$, for all $\heartsuit \in \mathbb{R}$.

$$\Rightarrow e^{-3x} = \sum_{n=0}^{\infty} \frac{(-3x)^n}{n!}, \forall x \in \mathbb{R}.$$

$$e^{-3x} \simeq \sum_{n=0}^{4} \frac{(-3x)^n}{n!} = 1 + (-3x) + \frac{(-3x)^2}{2!} + \frac{(-3x)^3}{3!} + \frac{(-3x)^4}{4!}$$

$$= 1 - 3x + \frac{9}{2}x^2 + \frac{-9}{2}x^3 + \frac{27}{8}x^4$$

$$f(0.12) = e^{-3(0.12)} \simeq 1 - 3(0.12) + 4.5(0.12)^2 - 4.5(0.12)^3 + \frac{27}{8}(0.12)^4$$

$$\simeq 0.69772384$$

Note: If we put in $y = e^{-3x}$ on calculator, we get 0.6976763261 at $x = 0.12$

Lagrange Error for Taylor Polynomials

We know $f(x) = P_n(x) + R_n(x)$.

error $\boxed{R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}$

$c \in (a,x)$ if $a < x$ (or $c \in (x, a)$ if $x < a$)

EX 2 Find the error in estimating $f(0.12)$ in the last example, $f(x) = e^{-3x}$.

since it's a $(a=0)$ Mclaurin series

error: $R_4(x) = \frac{f^{(5)}(c)}{5!}(x-0)^5$

$x = 0.12$ $\quad R_4(0.12) = \frac{f^{(5)}(c)}{5!}(0.12)^5$ $\quad c \in [0, 0.12]$

$f(x) = e^{-3x}$

$f'(x) = -3e^{-3x}$

$f''(x) = (-3)^2 e^{-3x}$

$f'''(x) = (-3)^3 e^{-3x}$

$\vdots$

$f^{(5)}(x) = (-3)^5 e^{-3x}$

$R_4(0.12) = \frac{(-3)^5 e^{-3c}(0.12)^5}{5!}$

$|R_4(0.12)| = \frac{3^5(0.12)^5}{5!} e^{-3c}$

$= \frac{0.36^5}{5!}(e^{-3c}) = \frac{0.36^5}{5!\, e^{3c}}$

monotonically decreasing

$c \in [0, 0.12]$

$\Rightarrow$ choose $c = 0$.

$\Rightarrow |R_4(0.12)| \le \frac{0.36^5}{5!}(e^{-3(0)}) = \frac{0.36^5}{5!} \approx 0.0000504$

EX 3 Find a good bound for the maximum value of $\left|\dfrac{4c}{c+4}\right|$ given $c \in [0,1]$.

$$\left|\frac{4c}{c+4}\right| = \frac{4c}{c+4}$$

(we want a reasonable upper bound.)

idea 1: look at numerator & denominator separately.

- $0 \le c \le 1 \Rightarrow 0 \le 4c \le 4 \Rightarrow 4c \le 4$
- $4 \le c+4 \le 5 \Rightarrow \frac{1}{4} \ge \frac{1}{c+4} \ge \frac{1}{5} \Rightarrow \frac{1}{c+4} \le \frac{1}{4}$

$$\Rightarrow \frac{4c}{c+4} = 4c\left(\frac{1}{c+4}\right) \le 4\left(\frac{1}{4}\right) = 1$$

idea 2:

$$\frac{4c}{c+4} = 4 - \frac{16}{c+4}$$

$$\begin{array}{r|l} & 4 - \frac{16}{c+4} \\ c+4 & 4c \\ & -(4c+16) \\ \hline & -16 \end{array}$$

$$\left|\frac{4c}{c+4}\right| = \left|4 - \frac{16}{c+4}\right| \le |4| + \left|\frac{16}{c+4}\right|$$

$$\le 4 + 16\left(\frac{1}{4}\right) = 4+4 = 8$$

I would choose idea 1.

EX 4 Find a good bound for the maximum value of $\left|\dfrac{c^2 - c}{\cos c}\right|$ given $c \in [0, \pi/4]$.

$$\left|\frac{c^2 - c}{\cos c}\right|$$

$$\frac{\sqrt{2}}{2} \le |\cos c| \le 1$$

$$\Rightarrow \frac{2}{\sqrt{2}} \ge \frac{1}{|\cos c|} \ge 1$$

$$\sqrt{2} \ge \frac{1}{|\cos c|} \ge 1$$

$c \in [0, \pi/4]$

upper bound for $\dfrac{1}{|\cos c|}$

$|c^2 - c|$

idea 1: use triangle inequality

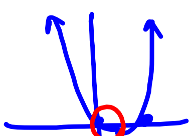
$$|a \pm b| \le |a| + |b|$$

$$\Rightarrow |c^2 - c| \le |c^2| + |c| \le \left(\frac{\pi}{4}\right)^2 + \frac{\pi}{4} \approx 1.4$$

idea 2:

look at $y = x^2 - x$

$x \in [0, \pi/4]$



we want an upper bound (max) of $y = x^2 - x$ on $[0, \pi/4]$

vertex is at $\left(\frac{1}{2}, -\frac{1}{4}\right)$ is the furthest distance from x-axis. on $[0, \pi/4]$

$$\Rightarrow |c^2 - c| \le \frac{1}{4}$$

$$\left|\frac{c^2 - c}{\cos c}\right| = |c^2 - c|\left|\frac{1}{\cos c}\right| \le \frac{1}{4}(\sqrt{2}) \approx 0.35355$$

EX 5 Find n such that the Maclaurin polynomial for $f(x) = e^x$ has $f(1)$ approximated to five decimal places, i.e. $|R_n(1)| \leq 0.000005$.

$$|R_n(1)| = \left| \frac{f^{(n+1)}(c)}{(n+1)!}(1)^n \right| \leq 0.000005$$

$c \in [0, 1]$

$$f^{(n+1)}(x) = e^x \Rightarrow \left| \frac{e^c}{(n+1)!} \right| \leq 0.000005$$

$e^0 \leq e^c \leq e^1 \Rightarrow$ upper bound is $e^c \leq e$

$$\frac{e}{(n+1)!} \leq 0.000005$$

$$\frac{e}{0.000005} \leq (n+1)!$$

$$\approx \quad 543{,}656.36 \leq (n+1)!$$

try $n=8$, not big enough

if $n=9$, $(n+1)! = 10! = 3{,}628{,}800$

$\Rightarrow$ we need Maclaurin polynomial of order 9.

Conclusion:

We're discussed bound on error of the method. It doesn't tell us anything about rounding errors.

ex $s = 1{,}000{,}000$

$s - a_1 - a_2 - a_3 - a_4 \cdots - a_n \qquad a_i = 0.001$

$1{,}000{,}000 - 0.001 - 0.001 - \cdots - 0.001$

If we do $(((s-a_1)-a_2)-a_3)\cdots - a_n)$, then we have too many error problems, and our answer will still be almost 1,000,000.

But if we do $s-(a_1+a_2+\cdots+a_n)$, then we'll get a closer answer. (avoid more errors this way).