If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

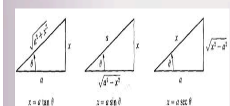
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

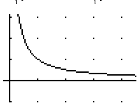
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad+\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x\frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt\cong 0.69315$$



$$\int u\,dv=uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

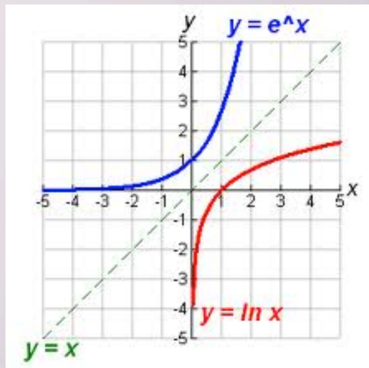
$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

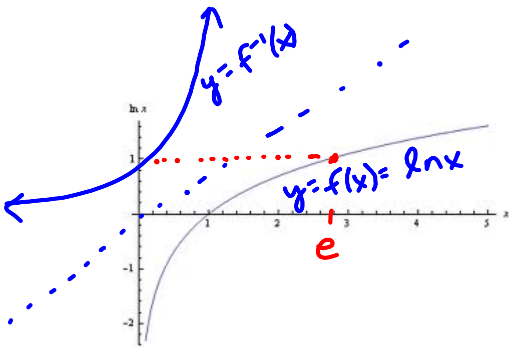
$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

The Natural Exponential Function



The Natural Exponential Function

Remember the graph of $y = \ln x$.



It is strictly monotonic, so it has an inverse function.

Draw it.

	$f(x)$	$f^{-1}(x)$
Domain:	$x > 0$ $(0, \infty)$	$x \in \mathbb{R}$ $(-\infty, \infty)$
Range:	$y \in \mathbb{R}$ $(-\infty, \infty)$	$y > 0$ $(0, \infty)$

domain of $f(x)$ = range of $f^{-1}(x)$.

Let's call the inverse function "exp."

because ln and exp are inverses

$ln(exp(x)) = x$

$exp(ln(x)) = x$

Definition: Let e be a real number such that $\ln e = 1$.

$r \in \mathbb{R}$, $\exp(r) = \exp(r \ln e)$ since $\ln e = 1$

$= \exp(\ln e^r)$

$= e^r$ $\Rightarrow$ very cool

we now know the inverse of $\ln(x) = f(x)$ is none other than our old friend the exponential fn w/ base e, i.e.

$f^{-1}(x) = e^x$

<u>Theorem</u> Let a and b be real numbers. Then $e^a e^b = e^{a+b}$ and $\frac{e^a}{e^b} = e^{a-b}$

<u>claim</u>: $\frac{e^a}{e^b} = e^{a-b}$

Proof:

$$\frac{e^a}{e^b} = \exp\left(\ln\left(\frac{e^a}{e^b}\right)\right)$$

$$= \exp(\ln e^a - \ln e^b)$$

$$= \exp(a \ln e - b \ln e)$$

$$= \exp(a \cdot 1 - b \cdot 1)$$

$$= \exp(a-b)$$

$$= e^{a-b}$$

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How do we take a derivative?

Let $y = e^x \iff \ln y = x$ (know the derivative of $\ln x$)

$D_x(\ln y) = D_x(x)$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = 1 \iff \frac{dy}{dx} = y = e^x$

$\Rightarrow \boxed{D_x(e^x) = e^x} \Rightarrow \boxed{\int e^x\,dx = e^x + C}$

EX 1 Find y'.

$y = e^{x^2-3x}$

$y' = e^{x^2-3x}(2x-3)$

EX 2 Find y'. $y = e^{\overset{1}{\sqrt{x}}\,\overset{2}{\ln x}}$

$$y' = e^{\sqrt{x}\ln x}\left(\tfrac{1}{2}x^{-1/2}\ln x + \sqrt{x}\left(\tfrac{1}{x}\right)\right)$$

$$= e^{\sqrt{x}\ln x}\left(\frac{\ln x}{2\sqrt{x}} + \frac{1}{\sqrt{x}}\right)$$

EX 3 For $f(x)$ analyze the graph. (I.e. min, max, concavity, inflection pts, sketch.)

$f(x) = e^{x} - e^{-x}$

$f(0) = e^{0} - e^{0} = 0$

$$f'(x) = e^{x} - e^{-x}(-1) = \underline{e^{x} + e^{-x}} = e^{x} + \frac{1}{e^{x}}$$

$$= \frac{e^{2x}+1}{e^{x}} > 0 \text{ always}$$

$\longleftarrow \overset{+}{\rule{3em}{0.4pt}} \longrightarrow f'(x)$ $\Rightarrow$ no min/max pts.

$$f''(x) = e^{x} + e^{-x}(-1) = \boxed{e^{x} - \frac{1}{e^{x}}} = 0$$

$$e^{x} = \frac{1}{e^{x}} \Leftrightarrow e^{2x} = 1$$

$$2x = 0$$

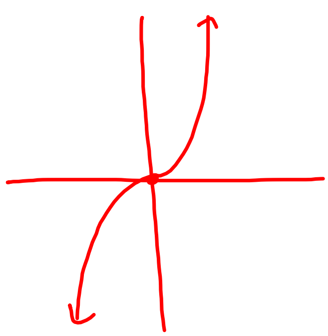
$$x = 0$$

$\longleftarrow - \mid + \longrightarrow f''(x)$

concave down 0 concave up

test: ① $x = -1$, $\frac{1}{e} - e < 0$

② $x = 1$, $e - \frac{1}{e} > 0$ $\Rightarrow$ inflectn pt at $(0, 0)$



Since $D_x\left[e^x\right] = e^x$, then $\int e^x dx = e^x + C$.

EX 4 Evaluate these integrals.

a) $\int e^{-6x} dx$ $= -\frac{1}{6}\int e^u \, du$

$u = -6x$

$du = -6\,dx$

$-\frac{1}{6}\,du = dx$

$= -\frac{1}{6}\left(e^u\right) + C$

$= \boxed{-\frac{1}{6} e^{-6x} + C}$

b) $\int e^{(x+e^x)} dx$

$= \int e^x e^{e^x} dx = \int e^u \, du$

$u = e^x$

$du = e^x dx$

$= e^u + C$

$= \boxed{e^{e^x} + C}$

c) $\int \frac{e^{3/x}}{x^2} dx$ $= -\frac{1}{3}\int e^u \, du$

$u = \frac{3}{x}$

$du = -3x^{-2} dx$

$-\frac{1}{3} du = \frac{1}{x^2} dx$

$= -\frac{1}{3} e^u + C$

$= \boxed{-\frac{1}{3} e^{3/x} + C}$

main point

$$\boxed{\begin{aligned} &D_x(e^x) = e^x \\ &\int e^x\,dx = e^x + C \end{aligned}}$$