If

$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$$

or

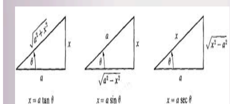
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$$

Then

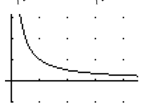
$$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2 \\ &\quad +\frac{f'''(x_0)}{3!}(x-x_0)^3+\frac{f''''(x_0)}{4!}(x-x_0)^4+\cdots \\ &= \sum_{n=0}^{\infty}\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n. \end{aligned}$$



$$\ln(x)=\int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2)=\int_1^2\frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv-\int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$$

put into reverse

$$\int\frac{d}{dx}(uv)=\int\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)$$

$$uv=\int u\frac{dv}{dx}+\int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx}=uv-\int v\frac{du}{dx}$$

General Logarithmic and Exponential Functions

1. $\dfrac{d}{dx}\left[a^x\right]=(\ln a)a^x$

2. $\dfrac{d}{dx}\left[a^u\right]=(\ln a)a^u\dfrac{du}{dx}$

General Exponential and Logarithmic Functions

$$a^x = \exp(\ln a^x) = \exp(x \ln a) = e^{x \ln a}$$
$$\Leftrightarrow$$
$$\ln(a^x) = \ln(e^{x \ln a}) = x \ln a$$

we can rewrite exponentials of any base in terms of base e exponential.

Properties of Exponents

$(i)\quad a^x a^y = a^{x+y}$

$(ii)\quad \dfrac{a^x}{a^y} = a^{x-y}$

$(iii)\quad (a^x)^y = a^{xy}$

$(iv)\quad (ab)^x = a^x b^x$

$(v)\quad \left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$

Pf
$$\left(\frac{a}{b}\right)^x = e^{\ln\left(\frac{a}{b}\right)^x}$$
$$= e^{x\ln\left(\frac{a}{b}\right)}$$
$$= e^{x(\ln a - \ln b)}$$
$$= e^{x\ln a - x\ln b}$$
$$= e^{\ln a^x - \ln b^x}$$
$$= e^{\ln\left(\frac{a^x}{b^x}\right)} = \frac{a^x}{b^x} \#$$

How do we find a derivative or an integral of a^x ?

we know $D_x(e^x)=e^x$

$D_x[a^x]= D_x(e^{\ln a^x}) = D_x(e^{x\ln a})$

(note: $\ln a$ is a constant)

$= e^{x \ln a}(\ln a)$

$= e^{\ln a^x}(\ln a) = a^x(\ln a)$

and

$\int a^x dx = \dfrac{a^x}{\ln a} + C$

$$\boxed{D_x\left[a^x\right] = a^x \ln a \qquad \int a^x dx = \frac{1}{\ln a}(a^x)+C \qquad a \neq 1}$$

$$D_x\left[a^x\right]=a^x \ln a$$

$$\int a^x dx=\frac{1}{\ln a}(a^x)+C \qquad a\neq 1$$

EX 1 Find y'.

$$y=\underbrace{\left(2x^3+9x\right)^4}_{\text{power term}}+\underbrace{4^{2x^3+9x}}_{\text{exponential term}}$$

$$y'=4(2x^3+9x)^3(6x^2+9)+4^{2x^3+9x}(\ln 4)\cdot(6x^2+9)$$

$$=(6x^2+9)\left[4(2x^3+9x)^3+(\ln 4)4^{2x^3+9x}\right]$$

EX 2 Evaluate $\int \frac{2^{\sqrt{x}}}{3\sqrt{x}}dx$.

$$=\frac{1}{3}\int \frac{2^{\sqrt{x}}}{\sqrt{x}}dx$$

$u=\sqrt{x}$

$du=\frac{1}{2}x^{-1/2}dx$

$2\,du=\frac{1}{\sqrt{x}}dx$

$$=\frac{1}{3}(2)\int 2^u\,du$$

$$=\frac{2}{3}\left(\frac{2^u}{\ln 2}\right)+C$$

$$=\boxed{\frac{2}{3\ln 2}\left(2^{\sqrt{x}}\right)+C}$$

or $\frac{2}{\ln 8}\left(2^{\sqrt{x}}\right)+C$

or $\frac{1}{\ln 8}\left(2^{\sqrt{x}+1}\right)+C$

Remember log definitions from algebra.

$y = log_a x \iff a^y = x$

$$\Updownarrow$$

$$\ln a^y = \ln x$$

$$y \ln a = \ln x$$

$$y = \boxed{\frac{\ln x}{\ln a} = \log_a x}$$ Change of base formula

$$\boxed{D_x(\log_a x)} = D_x\left(\frac{\ln x}{\ln a}\right) = \frac{1}{\ln a} D_x(\ln x) = \boxed{\frac{1}{\ln a}\left(\frac{1}{x}\right)}$$

We know $D_x x^a = a\,x^{a-1}$ is true for rational a. What if a is irrational?

(revisiting the power rule)

assume a is irrational.

$$D_x(x^a) = D_x\left(e^{\ln x^a}\right)$$

$$= D_x\left(e^{a\ln x}\right)$$

$$= e^{a\ln x}\left(a\left(\frac{1}{x}\right)\right)$$

$$= x^a\left(\frac{a}{x}\right) = a x^{a-1}$$

#

EX 3 Find y'. $y = \sin^2 x + 2^{\sin x}$

$y = (\sin x)^2 + 2^{\sin x}$

power rule/ chain rule

exponential chain rule

$D_x(a^x) = a^x \ln a$

$y' = 2(\sin x)^1(\cos x) + 2^{\sin x}(\ln 2)(\cos x)$

$y' = \cos x \left(2 \sin x + (\ln 2)\, 2^{\sin x}\right)$

EX 4 Find y'. $y = x^x$ Hint: Take the log of both sides.

note: we can't "classify" this fn.

use (logarithmic differentiation)

$\ln y = \ln x^x$

D_x: $\ln y = \underbrace{x}_{1}\underbrace{\ln x}_{2}$

$\frac{1}{y}\left(\frac{dy}{dx}\right) = 1 \cdot \ln x + x\left(\frac{1}{x}\right)$

$\frac{1}{y}(y') = \ln x + 1 \Rightarrow y' = y(\ln x + 1) = \boxed{x^x(\ln x + 1)}$

EX 5 Evaluate $\int_0^1 \left(10^{3x} + 10^{-3x}\right) dx$.

need: $a>0, a\neq 1$

$\int a^x dx = \frac{a^x}{\ln a} + C$

$= \int_0^1 10^{3x} dx + \int_0^1 10^{-3x} dx$

$= \int_0^1 1000^x dx + \int_0^1 \left(\frac{1}{1000}\right)^x dx$

$10^{3x} = (10^3)^x = 1000^x$

$= \frac{1000^x}{\ln 1000}\Big|_0^1 + \frac{\left(\frac{1}{1000}\right)^x}{\ln\left(\frac{1}{1000}\right)}\Bigg|_0^1$

$= \left(\frac{1000}{\ln 1000} - \frac{1}{\ln 1000}\right) + \left(\frac{\frac{1}{1000}}{\ln\left(\frac{1}{1000}\right)} - \frac{1}{\ln\left(\frac{1}{1000}\right)}\right)$

$= \frac{999}{\ln 1000} + \frac{-999}{1000}\left(\frac{1}{-\ln 1000}\right)$

$\ln\left(\frac{1}{1000}\right) = \ln 1000^{-1} = -\ln 1000$

$= \frac{1}{\ln 1000}\left(999 + \frac{999}{1000}\right)$

$= \boxed{\frac{999.999}{\ln 1000}}$

EX 6 If $y=(\ln x^2)^{2x+3}$ find y'.

note:

$(\ln x^2)^{(2x+3)}$

$= (\ln(x^2))^{2x+3}$

$$\ln y = \ln(\ln x^2)^{2x+3}$$

$$D_x: \quad \ln y = \underbrace{(2x+3)}_{1} \underbrace{\ln(\ln x^2)}_{2}$$

$$\frac{1}{y}y' = 2\left(\ln(\ln x^2)\right) + (2x+3)\left(\frac{1}{\ln x^2}\right)\left(\frac{1}{x^{\cancel{2}}}\right)(2\cancel{x})$$

$$y' = y\left[2\ln(\ln x^2) + \frac{\cancel{2}(2x+3)}{\cancel{x}\,x\ln x^2}\right]$$

$$y' = (\ln x^2)^{2x+3}\left[2\ln(\ln x^2) + \frac{2x+3}{x\ln x}\right]$$

Final Message

$a > 0, a \neq 1$

$$D_x\left(a^x\right) = (\ln a)\, a^x \qquad D_x(\log_a x) = \frac{1}{x \ln a}$$

$$\int a^x\,dx = \frac{1}{\ln a} a^x + C$$