If

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0}$$

or

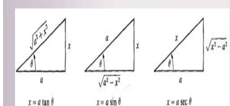
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

Then

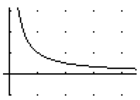
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided that the latter limit exists.

$$\begin{aligned} f(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ &+ \frac{f'''(x_0)}{3!}(x - x_0)^3 + \frac{f''''(x_0)}{4!}(x - x_0)^4 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n. \end{aligned}$$



$$\ln(x) = \int_1^x \frac{1}{t}\,dt \Rightarrow \ln(2) = \int_1^2 \frac{1}{t}\,dt \cong 0.69315$$



$$\int u\,dv = uv - \int v\,du$$

where it comes from: the product rule for differentiation

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

put into reverse

$$\int \frac{d}{dx}(uv) = \int \left(u\frac{dv}{dx} + v\frac{du}{dx}\right)$$

$$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$$

and then rearranged

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$$

Integration by Parts

$$\int udv = uv - \int vdu$$

Use the product rule for differentiation

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

Integrate both sides

$$\int \frac{d}{dx}(uv) = \int \left(u\frac{dv}{dx} + v\frac{du}{dx}\right)$$

Simplify

$$uv = \int u\frac{dv}{dx} + \int v\frac{du}{dx}$$

Rearrange

$$\int u\frac{dv}{dx} = uv - \int v\frac{du}{dx}$$

Integration by Parts

Look at the Product Rule for Differentiation.

$u = u(x)$, $v = v(x)$

$$D_x[u(x)v(x)] = u'(x)v(x) + v'(x)u(x)$$

$$D_x[uv] = u'v + v'u$$

$$D_x[uv] - u'v = uv'$$

$$uv' = D_x[uv] - u'v$$

$$\int uv'\,dx = \int \left(D_x[uv] - u'v \right) dx$$

$v' = \frac{dv}{dx}$

$$\int u\left(\frac{dv}{dx}\right)dx = \int \frac{d[uv]}{dx}\,dx - \int \left(\frac{du}{dx}\right)v\,dx$$

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Integration by Parts formula

Notes: • $u \cdot dv$ must account for the entire integrand

• this is like double u-sub.

$$\int u\,dv = uv - \int v\,du$$

EX 1 $\int x\sin(2x)\,dx$ ($u = x$, $dv = \sin(2x)\,dx$)

$u = x$ $\quad v = \int \sin(2x)\,dx = -\frac{1}{2}\cos(2x)$

$du = dx$ $\quad dv = \sin(2x)\,dx$

$$= x\left(-\frac{1}{2}\cos(2x)\right) - \int -\frac{1}{2}\cos(2x)\,dx$$

$$= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\int \cos(2x)\,dx$$

$$= -\frac{1}{2}x\cos(2x) + \frac{1}{2}\left(\frac{1}{2}\right)\sin(2x) + C$$

$$= -\frac{1}{2}x\cos(2x) + \frac{1}{4}\sin(2x) + C$$

Classic "Integration by Parts" Integrals

1. a polynomial times a sine or cosine fn
2. exponential times sine or cosine fn
3. polynomial times an exponential fn
4. a fn that we don't know how to integrate but we do know how to differentiate

EX 2 $\int \underbrace{\arctan(5x)}_{u}\, \underbrace{dx}_{dv}$

$$\int u\,dv = uv - \int v\,du$$

Int by Parts

$u = \arctan(5x)$ $\quad v = x$

$du = \frac{1(5)}{1+(5x)^2}dx$ $\quad dv = dx$

$du = \frac{5}{1+25x^2}dx$

$= x\arctan(5x) - \int x\left(\frac{5}{1+25x^2}\right)dx$

$= x\arctan(5x) - \int \frac{5x}{1+25x^2}dx$

$u = 1+25x^2$

$du = 50x\,dx$

$\frac{1}{10}du = 5x\,dx$

$= x\arctan(5x) - \frac{1}{10}\int \frac{1}{u}\,du$

$= x\arctan(5x) - \frac{1}{10}\ln|u| + C$

$= \boxed{x\arctan(5x) - \frac{1}{10}\ln(1+25x^2) + C}$

EX 3 $\int \frac{\ln x}{\sqrt{x}}\,dx$

① Try

$u = \ln x$

$du = \frac{1}{x}\,dx$

doesn't work

② Try Int. by Parts

$u = \ln x \qquad v = 2x^{1/2}$

$du = \frac{1}{x}\,dx \qquad dv = \frac{1}{\sqrt{x}}\,dx \quad (= x^{-1/2}\,dx)$

$= 2x^{1/2}\ln x - \int 2x^{1/2}\left(\frac{1}{x}\right)dx$

$= 2\sqrt{x}\ln x - 2\int x^{-1/2}\,dx$

$= 2\sqrt{x}\ln x - 2(2x^{1/2}) + C$

$= \boxed{2\sqrt{x}\ln x - 4\sqrt{x} + C}$

Repeated Integration by Parts

EX 4 $\int \underbrace{x^3}_{u}\underbrace{e^x dx}_{dv}$ $= x^3e^x - 3\int \underbrace{x^2}_{u}\underbrace{e^x dx}_{dv}$

Same process for $\int x^3 \cos x\, dx$ or $\int x^3 \sin x\, dx$

$u = x^3$ $\quad v = e^x$

$du = 3x^2 dx$ $\quad dv = e^x dx$

$u = x^2$ $\quad v = e^x$

$du = 2x\, dx$ $\quad dv = e^x dx$

$= x^3e^x - 3\left(x^2e^x - 2\int xe^x dx\right)$

$= x^3e^x - 3x^2e^x + 6\int xe^x dx$

$u = x$ $\quad v = e^x$

$du = dx$ $\quad dv = e^x dx$

$= x^3e^x - 3x^2e^x + 6\left(xe^x - \int e^x dx\right)$

$= \boxed{x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C}$

EX 5 $\int e^x \cos x \, dx$

($u = e^x$, $dv = \cos x\,dx$)

$u = e^x \qquad v = \sin x$

$du = e^x dx \qquad dv = \cos x \, dx$

$= e^x \sin x - \int e^x \sin x \, dx = e^x \sin x - \left(-e^x \cos x - \int -e^x \cos x \, dx\right)$

$u = e^x \qquad v = -\cos x$

$du = e^x dx \qquad dv = \sin x \, dx$

$= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$

Here's what we have now

$$\int e^x \cos x \, dx = e^x \sin x + e^x \cos x - \int e^x \cos x \, dx$$

$+\int e^x \cos x \, dx \qquad\qquad + \int e^x \cos x \, dx$

$$\frac{2\int e^x \cos x \, dx}{2} = \frac{e^x \sin x + e^x \cos x}{2}$$

$$\boxed{\int e^x \cos x \, dx = \frac{1}{2} e^x (\sin x + \cos x) + C}$$

Conclusion

Integration By Parts

$$\int u\,dv = uv - \int v\,du$$