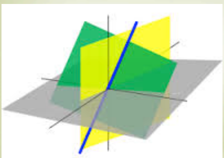
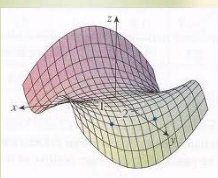
Maxima and Minima



$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

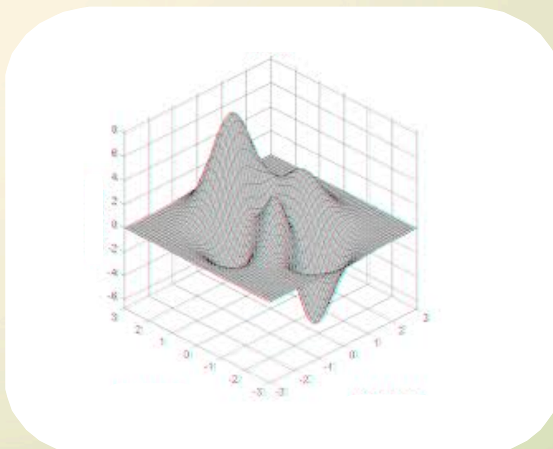
$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$

$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

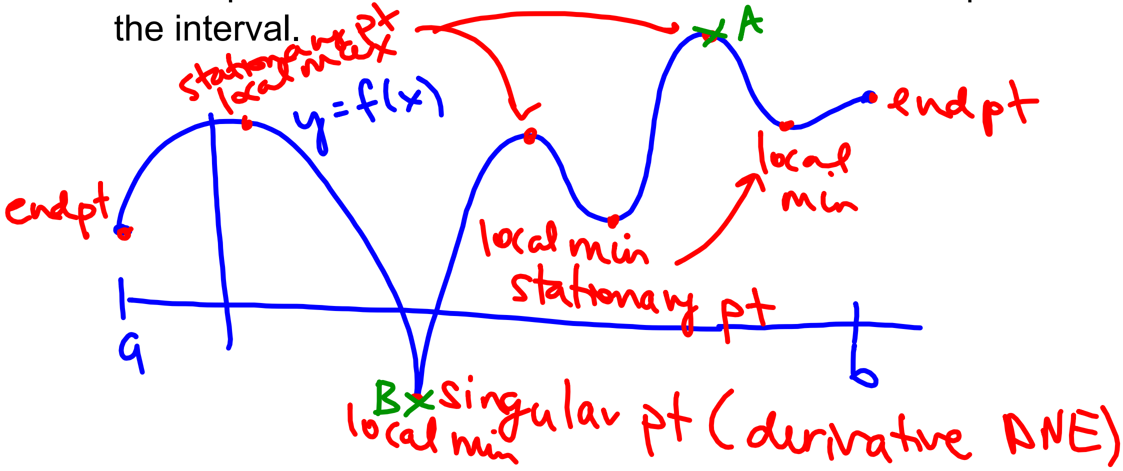


Recall from Calculus I: (curves in 2-d)

1) Critical points (where $f'(x) = 0$ or DNE) are the candidates for where local min and max points can occur.

2) You can use the Second Derivative Test (SDT) to test whether a given critical point is a local min or max. SDT is not always conclusive.

3) Global max and min of a function on an interval can occur at a critical point in the interior of the interval or at the endpoints of the interval.

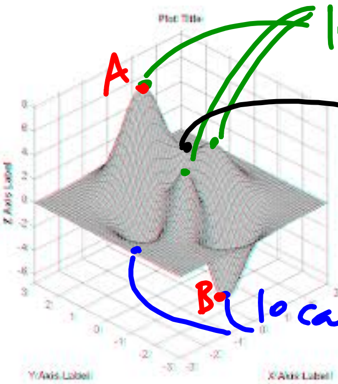


global max occurs at pt A
global min " " " B } on $[a, b]$

Extreme Values

1) f has a global maximum at a point (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) in the domain of f. f has a local maximum at a point (a,b) if $f(a,b)) \geq f(x,y)$ for all (x,y) near (a,b).

2) f has a global minimum at a point (a,b) if $f(a,b) \leq f(x,y)$ for all (x,y) in the domain of f. f has a local minimum at a point (a,b) if $f(a,b)) \leq f(x,y)$ for all (x,y) near (a,b).



local max pts ("peak pts")

saddle pt (surface goes up in some direction but down in another)

local min pt ("valley pts")

global min/max pts:

global max at A

global min at B

Theorem (Critical Point)

Let f be defined on a set S containing (a,b). If $f\,(a,b)$ is an extreme value (max or min),

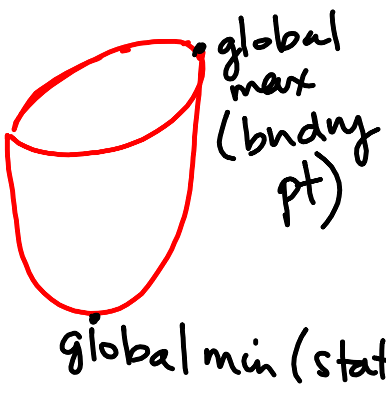
then (a,b) must be a critical point, i.e. either (a,b) is

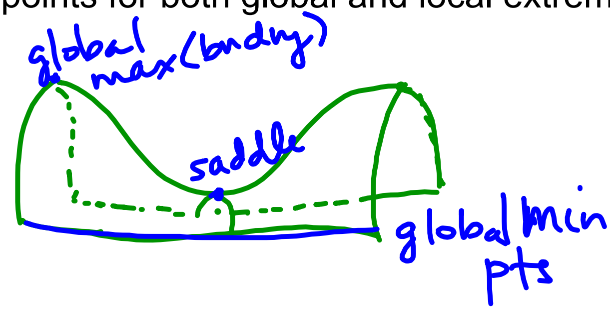
a) a boundary point of S (assumes S is closed + bounded)

b) a stationary point of S (where $\nabla f\,(a,b) = \vec{0}$, i.e. the tangent plane is horizontal)

c) a singular point of S (where f is not differentiable).

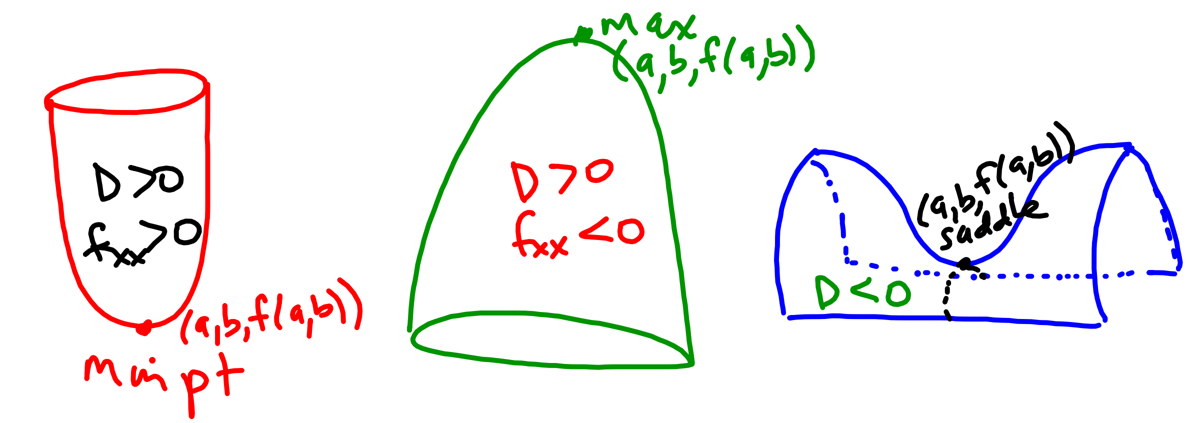
Fact: Critical points are candidate points for both global and local extrema.





Theorem (Max-Min Existence)

If f is continuous on a closed, bounded set S, then f attains both a global max value and a global min value there.



Second Partials Test Theorem

Suppose $f(x,y)$ has continuous second partial derivatives in a neighborhood of (a,b) and $\nabla f(a,b) = \vec{0}$.

Let $D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - f_{xy}^2(a,b)$

then

$D>0$

1) If $D>0$ *and* $\underline{f_{xx}(a,b)<0}$, then $f(a,b)$ is a local max.
2) If $D>0$ *and* $\underline{f_{xx}(a,b)>0}$, then $f(a,b)$ is a local min.

$f_{xx}(a,b)$ and $f_{yy}(a,b)$ are same sign

3) If $D<0$, then $f(a,b)$ is not an extreme value.

   ((a,b) is a saddle point.)

4) If $D = 0$ the test is inconclusive.

EX 1 For $f(x,y) = xy^2 - 6x^2 - 3y^2$, find all critical points,

indicating whether each is a local min, a local max or saddle point.

$f_x = y^2 - 12x$ $\qquad f_y = 2xy - 6y$

$f_{xx} = -12$, $f_{yy} = 2x - 6$, $f_{xy} = 2y$

$D = f_{xx} f_{yy} - f_{xy}^2 = -12(2x-6) - (2y)^2$

$= -24x + 72 - 4y^2$

possible stationary / singular pts:

$\nabla f = \langle y^2 - 12x, 2xy - 6y \rangle = \langle 0, 0 \rangle$ (possible stationary pts)

note: no singular pts (∇f well-defined everywhere)

① $y^2 - 12x = 0$ and ② $2xy - 6y = 0$

$y^2 = 12x$

$x = \frac{y^2}{12} \longrightarrow 2\left(\frac{y^2}{12}\right)y - 6y = 0$

$\frac{1}{6}y^3 - 6y = 0$

$y(y^2 - 36) = 0$

$y = 0, 6, -6$

$\Rightarrow$ if $y = 0$, $x = \frac{0^2}{12} = 0$ $\quad A(0,0)$

if $y = \pm 6$, $x = \frac{(\pm 6)^2}{12} = 3$ $\quad B(3,6)$ $C(3,-6)$

$D = -24x + 72 - 4y^2$, $f_{xx} = -12$

pt A: $D = -24(0) + 72 - 4(0) = 72 > 0$

$f_{xx}(0,0) = -12 < 0 \Rightarrow \boxed{\text{max pt}}$

pt B: $D = -24(3) + 72 - 4(36) = 0 - 4(36) < 0$

$\Rightarrow \boxed{\text{saddle pt}}$

pt C: $D = -24(3) + 72 - 4(36) < 0$

$\Rightarrow \boxed{\text{saddle pt}}$

no local min pt

max pt: $(0,0,0)$

saddle pts: $(3,6,-54)$

$(3,-6,-54)$

$z = xy^2 - 6x^2 - 3y^2$

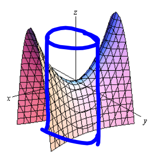
$z = 3(\cancel{6^2}) - 6(3^2) - 3(\cancel{6^2}) = -54$

EX 2 Find the global max and min values for

$f(x,y) = x^2 - y^2 - 1$ on

$\underbrace{S = \{(x,y) \mid x^2 + y^2 \le 1\}}_{\text{bndry}}$

(consider local min/max pts and bndry pts)



hyperbolic paraboloid

$f_x = 2x\ ,\quad f_y = -2y$

$\nabla f = \langle 2x, -2y \rangle = \langle 0, 0 \rangle$

$x = 0$, and $y = 0$ stationary pt to consider

this turns out to be a saddle pt

note: no singular pts

boundary pts: $\{(x,y) \mid x^2 + y^2 = 1\}$

by inspection: we can see that max pts occur along x-axis and min pts occur along y-axis.

along x-axis: $y = 0$, $x^2 + 0^2 = 1 \Rightarrow x = \pm 1$

$f(\pm 1, 0) = (\pm 1)^2 - 0^2 - 1 = 0$

$\boxed{\text{max pts } (\pm 1, 0, 0)}$

along y-axis: $x = 0$, $0^2 + y^2 = 1 \Rightarrow y = \pm 1$

$f(0, \pm 1) = 0^2 - (\pm 1)^2 - 1 = -2$

$\boxed{\text{min pts } (0, \pm 1, -2)}$

Similarly, we can argue:

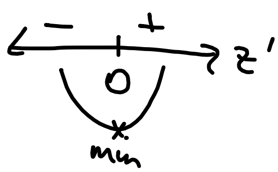
$f(x,y) = x^2 - y^2 - 1$ w/ bndry info $y^2 = 1 - x^2$

$z = f(x,y) = x^2 - (1 - x^2) - 1 = 2x^2 - 2$ (a fn of x only)

$z = 2x^2 - 2$

$\frac{dz}{dx} = z' = 4x = 0 \Leftrightarrow x = 0$

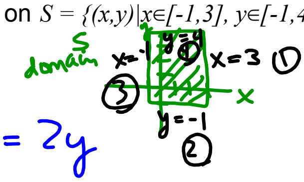
min pt at $(0, \pm 1, -2)$

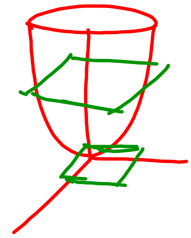


EX 3 Find the points where the global max and min occur for

$z = f(x,y) = x^2 + y^2$ on $S = \{(x,y) | x \in [-1,3], y \in [-1,4]\}$.

paraboloid





$f_x = 2x$, $f_y = 2y$

$\nabla f = \langle 2x, 2y \rangle = \langle 0, 0 \rangle \Leftrightarrow x = 0$ & $y = 0$

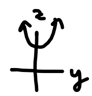
$\boxed{(0,0,0) \text{ min pt. (global)}}$

max pt is somewhere on the bndy

case ① $x = 3$:

$z = 3^2 + y^2$

$z = 9 + y^2$



(want to look for max pts)

$y \in [-1, 4]$

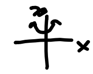
$\Rightarrow$ max pt occurs at bndy pt (y, z)

$(-1, 10)$ $z = 9 + 1$

$(4, 25)$ $z = 9 + y^2$

case ② $y = -1$

$z = x^2 + 1$



max pt will be a bndy pt; bndy $x \in [-1, 3]$

$(-1, 2)$ $z = 1 + 1$

$(3, 10)$ $z = 9 + 1$

case ③ $x = -1$

$z = 1 + y^2$ $y \in [-1, 4]$

bndy pts (y, z)

$(-1, 2)$ $z = 1 + 1$

$(4, 17)$ $z = 1 + 16$

case ④ $y = 4$

$z = 16 + x^2$ $x \in [-1, 3]$

bndy pts (x, z)

$(-1, 17)$ $z = 16 + 1$

$(3, 25)$ $z = 16 + 9$

possible max pts: (on surface)

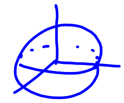
① $(3, -1, 10)$, $(3, 4, 25)$ ② $(-1, -1, 2)$, $(3, -1, 10)$ ③ $(-1, -1, 2)$, $(-1, 4, 17)$

④ $(-1, 4, 17)$, $(3, 4, 25)$

$\Rightarrow$ $\boxed{\text{global max occurs at } (3, 4, 25)}$

EX 4 Find the 3-D vector of length *9* with the largest possible sum of its components.

$x^2+y^2+z^2=9^2$



because of symmetry, our pt will be in 1st octant.

$f=x+y+z \Rightarrow f(x,y)=x+y+\sqrt{81-x^2-y^2}$

(domain $x^2+y^2\le 81$)

$\nabla f=\left\langle 1+\frac{-2x}{2\sqrt{81-x^2-y^2}},\ 1+\frac{-2y}{2\sqrt{81-x^2-y^2}}\right\rangle=\langle 0,0\rangle$

① $1+\frac{-\cancel{2}x}{\cancel{2}\sqrt{81-x^2-y^2}}=0$ and ② $1+\frac{-2y}{2\sqrt{81-x^2-y^2}}=0$

$81-x^2-y^2=x^2$ $\qquad$ $81=2y^2+x^2$

$81=2x^2+y^2$

$\Rightarrow 2x^2+y^2=2y^2+x^2$

2a $x^2=y^2$

① $81=2x^2+x^2$ (substitute $y^2=x^2$)

$x^2=27 \Rightarrow x=\pm 3\sqrt{3} \overset{\text{choose}}{\Rightarrow} x=3\sqrt{3}$

(to be in octant 1)

2a $y=3\sqrt{3}$

$f(3\sqrt{3},3\sqrt{3})=3\sqrt{3}+3\sqrt{3}+\sqrt{81-27-27}$

$=3\sqrt{3}+3\sqrt{3}+3\sqrt{3}=9\sqrt{3}$

note: if we check boundary pts $x^2+y^2=81$ we get possible max pts at

$(x,y)=(\pm 9/\sqrt{2},\pm 9/\sqrt{2}) \Rightarrow f(\pm 9/\sqrt{2},\pm 9/\sqrt{2})=0$ or $9\sqrt{2}$

both those f-values $<9\sqrt{3}$

$\Rightarrow$ max truly occurs at $(x,y)=(3\sqrt{3},3\sqrt{3})$

3-d vector we want: $\langle x,y,z\rangle$ where $z=\sqrt{81-x^2-y^2}$

is $\boxed{\langle 3\sqrt{3},3\sqrt{3},3\sqrt{3}\rangle}$