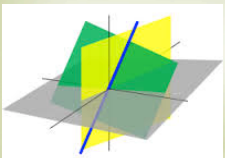
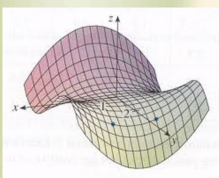
Lagrange Multipliers

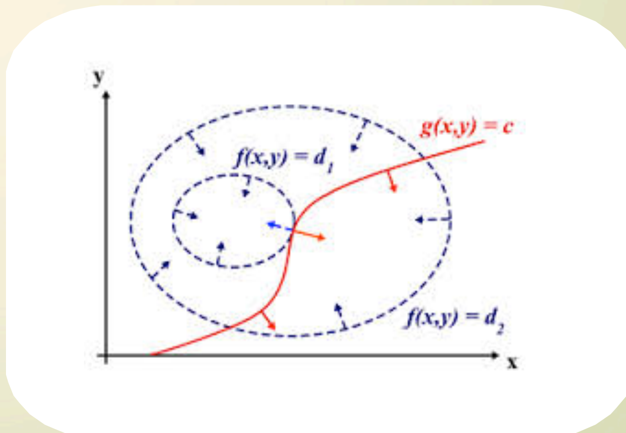


$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy = \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy = \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$



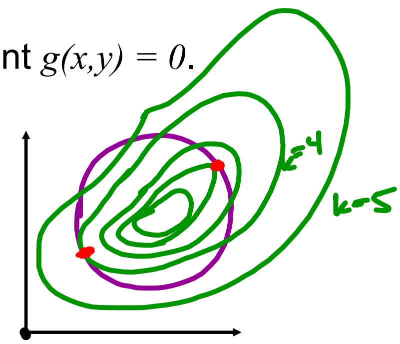
Now we will see an easier way to solve extrema problems with some constraints.

We want to optimize $f(x,y)$ subject to constraint $g(x,y) = 0$.

Graphically:

: level curves $(f(x,y) = k)$

: constraint curve



To maximize f subject to $g(x,y) = 0$ means to find the level curve of f with greatest k-value that intersects the constraint curve. It will be the place where the two curves are tangent.

Two curves have a common perpendicular line if they are tangent at that point. We know ∇f is perpendicular to its level curves. ∇g is also perpendicular to the constraint curve.

$\Rightarrow$ we want the pts where

$$\nabla f = c\nabla g$$

Theorem (Lagrange's Method)

To maximize or minimize $f(x,y)$ subject to constraint $g(x,y)=0$, solve the system of equations

① $\nabla f(x,y) = \lambda \nabla g(x,y)$ and ② $g(x,y) = 0$

for (x,y) and λ. The solutions (x,y) are critical points for the constrained extremum problem and the corresponding λ is called the Lagrange Multiplier.

Note: Each critical point we get from these solutions is a candidate for the max/min.

EX 1 Find the maximum value of $f(x,y) = xy$ subject to the constraint

$g(x,y) = 4x^2 + 9y^2 - 36 = 0.$

① $\nabla f = \lambda \nabla g \qquad \langle y, x\rangle = \lambda\langle 8x, 18y\rangle$

(1A) $y = 8\lambda x \qquad$ (1B) $x = 18\lambda y$

② $4x^2+9y^2-36=0$

1A $\Rightarrow$ 1B $\quad x = 18\lambda(8\lambda x)$

$x(1-144\lambda^2)=0$

$x=0$ or $1=144\lambda^2$

$\overset{1A}{\Rightarrow} y=0 \qquad \lambda = \pm 1/12$

$\overset{②}{\Rightarrow} 0+0-36\neq 0$

$\Rightarrow$ N.S.

$\lambda = \frac{1}{12}$:

(1A) $\Rightarrow y = \frac{8}{12}x = \frac{2}{3}x$

② $\Rightarrow 4x^2+9\left(\frac{2}{3}x\right)^2=36$

$8x^2=36$

$x^2=\frac{9}{2}$

$x=\pm\frac{3}{\sqrt{2}}$

$\Rightarrow x=\frac{3}{\sqrt{2}}, y=\frac{2}{3}\left(\frac{3}{\sqrt{2}}\right)$

$x=\frac{-3}{\sqrt{2}}, y=\frac{2}{3}\left(\frac{-3}{\sqrt{2}}\right)=-\sqrt{2}$

$\boxed{\lambda=\frac{1}{12},\ (x,y)=\left(\frac{3}{\sqrt{2}},\sqrt{2}\right),\ (x,y)=\left(\frac{-3}{\sqrt{2}},-\sqrt{2}\right)}$

$\lambda = \frac{-1}{12}$: (1A) $\Rightarrow y = \frac{-8}{12}x = \frac{-2}{3}x$

② $\Rightarrow 4x^2+9\left(\frac{-2}{3}x\right)^2=36$

$\vdots$

$x=\frac{\pm 3}{\sqrt{2}}$

$x=\frac{3}{\sqrt{2}}, y=\frac{-2}{3}\left(\frac{3}{\sqrt{2}}\right)=-\sqrt{2}$

or $x=\frac{-3}{\sqrt{2}}, y=\frac{-2}{3}\left(\frac{-3}{\sqrt{2}}\right)=\sqrt{2}$

$\boxed{\lambda=\frac{-1}{12},\ (x,y)=\left(\frac{3}{\sqrt{2}},-\sqrt{2}\right),\ (x,y)=\left(\frac{-3}{\sqrt{2}},\sqrt{2}\right)}$

critical pts : $z=f(x,y)=xy$

$\left(\frac{3}{\sqrt{2}}, -\sqrt{2}, -3\right)$

$\left(\frac{3}{\sqrt{2}}, \sqrt{2}, 3\right)$

$\left(\frac{-3}{\sqrt{2}}, -\sqrt{2}, 3\right)$

$\left(\frac{-3}{\sqrt{2}}, \sqrt{2}, -3\right)$

$\Rightarrow$ global min at $\left(\frac{3}{\sqrt{2}}, -\sqrt{2}, -3\right)$ and $\left(\frac{-3}{\sqrt{2}}, \sqrt{2}, -3\right)$

$\Rightarrow$ global max at $\left(\frac{-3}{\sqrt{2}}, -\sqrt{2}, 3\right)$ and $\left(\frac{3}{\sqrt{2}}, \sqrt{2}, 3\right)$

EX 2 Find the least distance between the origin and the plane

$\underbrace{x + 3y - 2z = 4}_{\text{constraint}}$. $g(x,y,z)$

(Note: minimizing the squared distance necessarily minimizes the distance)

$d = \sqrt{x^2+y^2+z^2}$

$f(x,y,z) = x^2+y^2+z^2$

② $g(x,y,z) = x+3y-2z-4=0$

① $\nabla f = \lambda \nabla g$

$\langle 2x, 2y, 2z \rangle = \lambda \langle 1, 3, -2 \rangle$

1A $2x=\lambda$ 1B $2y = 3\lambda$ 1C $2z=-2\lambda$

$x = \frac{\lambda}{2}$ $y = \frac{3}{2}\lambda$ $z = -\lambda$

$\Rightarrow$ ② $\frac{\lambda}{2} + 3\left(\frac{3}{2}\lambda\right) - 2(-\lambda) - 4 = 0$

$\frac{\lambda}{2} + \frac{9}{2}\lambda + 2\lambda = 4$

$7\lambda = 4$

$\lambda = 4/7$

$\Rightarrow x = \frac{1}{2}\left(\frac{4}{7}\right) = \frac{2}{7}$ $z = \frac{-4}{7}$

$y = \frac{3}{2}\left(\frac{4}{7}\right) = \frac{6}{7}$

note:

greatest distance from (0,0,0) to any plane $\rightarrow \infty$.

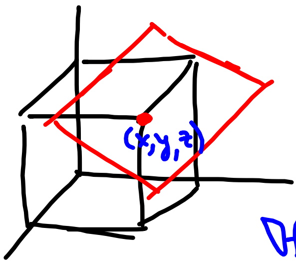
$\Rightarrow$ the pt we found is the least distance from origin.

pt is at $\left(\frac{2}{7}, \frac{6}{7}, \frac{-4}{7}\right)$

$\Rightarrow$ distance from origin $= \sqrt{\left(\frac{2}{7}\right)^2 + \left(\frac{6}{7}\right)^2 + \left(\frac{-4}{7}\right)^2}$

$= \frac{\sqrt{56}}{7} = \frac{2\sqrt{14}}{7}$

EX 3 Find the max volume of the first-octant rectangular box (with faces parallel to coordinate planes) with one vertex at *(0,0,0)* and the diagonally opposite vertex on the plane $\underbrace{3x + y + 2z = 1}_{\text{constraint}}$.



volume

$f(x,y,z) = xyz$

② $g(x,y,z) = 3x + y + 2z - 1 = 0$

$\nabla f = \lambda \nabla g$

① $\langle yz, xz, xy \rangle = \lambda \langle 3, 1, 2 \rangle$

1a $yz = 3\lambda$ 1b $xz = \lambda$ 1c $xy = 2\lambda$

$y = \frac{3\lambda}{z}$ $x = \frac{\lambda}{z}$ $\Rightarrow \left(\frac{\lambda}{z}\right)\left(\frac{3\lambda}{z}\right) = 2\lambda$

(assume $z \neq 0$)

case 1: $\lambda = 0$ case 2: $\lambda = \frac{2}{3}z^2$

case 1: $\lambda = 0$

$y = 0, x = 0,$ ② $3(0) + 0 + 2z = 1$

$z = 1/2$

$(0, 0, 1/2) \Rightarrow V = 0$

min

case 2: $\lambda = \frac{2}{3}z^2$

1a $\Rightarrow y = \frac{3}{z}\left(\frac{2}{3}z^2\right) = 2z$

1b $\Rightarrow x = \frac{1}{z}\left(\frac{2}{3}z^2\right) = \frac{2}{3}z$

② $3\left(\frac{2}{3}z\right) + 2z + 2z = 1$

$6z = 1$

$z = \frac{1}{6}$

$x = \frac{2}{3}\left(\frac{1}{6}\right) = \frac{1}{9}$

$y = 2\left(\frac{1}{6}\right) = \frac{1}{3}$

$\lambda = \frac{2}{3}\left(\frac{1}{6}\right)^2 = \frac{1}{54}$ max

$\Rightarrow V = \frac{1}{9}\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) = \frac{1}{162}$

If we have more than one constraint, additional Lagrange multipliers are used. If we want to maiximize $f(x,y,z)$ subject to $g(x,y,z)=0$ and $h(x,y,z)=0$, then we solve

$$\boxed{\nabla f = \lambda\nabla g + \mu\nabla h \text{ with } g=0 \text{ and } h=0\,.}$$

EX 4 Find the minimum distance from the origin to the line of intersection of the two planes.

$x + y + z = 8$ and $2x - y + 3z = 28$

$w = f(x,y,z) = x^2+y^2+z^2$

(2) $g(x,y,z) = x+y+z-8=0$

(3) $h(x,y,z) = 2x-y+3z-28=0$

$\nabla f = \lambda\nabla g + \mu\nabla h$

(1) $\langle 2x, 2y, 2z\rangle = \lambda\langle 1,1,1\rangle + \mu\langle 2,-1,3\rangle$

(1a) $2x = \lambda + 2\mu$ (1b) $2y = \lambda - \mu$ (1c) $2z = \lambda + 3\mu$

$x = \frac{1}{2}\lambda + \mu$ $y = \frac{1}{2}\lambda - \frac{1}{2}\mu$ $z = \frac{1}{2}\lambda + \frac{3}{2}\mu$

(2) $\frac{1}{2}\lambda + \mu + \frac{1}{2}\lambda - \frac{1}{2}\mu + \frac{1}{2}\lambda + \frac{3}{2}\mu = 8$

$\frac{3}{2}\lambda + 2\mu = 8 \iff 3\lambda + 4\mu = 16$

(3) $2(\frac{1}{2}\lambda + \mu) - (\frac{1}{2}\lambda - \frac{1}{2}\mu) + 3(\frac{1}{2}\lambda + \frac{3}{2}\mu) = 28$

$2\lambda + 7\mu = 28$

$-2(3\lambda + 4\mu = 16)$
$3(2\lambda + 7\mu = 28)$
$\iff$
$$\begin{array}{r} -6\lambda - 8\mu = -32 \\ + \quad 6\lambda + 21\mu = 84 \\ \hline 13\mu = 52 \\ \boxed{\mu = 4} \end{array}$$

$\Rightarrow 3\lambda + 4(4) = 16$

$3\lambda = 0$

$\boxed{\lambda = 0}$

$\Rightarrow x = \frac{1}{2}(0) + 4 = 4$

$y = \frac{1}{2}(0) - \frac{1}{2}(4) = -2$

$z = \frac{1}{2}(0) + \frac{3}{2}(4) = 6$

(note: max distance $\to \infty$)

$\Rightarrow$ min distance at $(4,-2,6)$

min distance of $\sqrt{4^2 + (-2)^2 + 6^2} = \sqrt{16+4+36}$

$= \sqrt{56} = \boxed{2\sqrt{14}}$

Lagrange multipliers don't work well for constraint regions like a square or triangle because there is not one equation to represent $g(x,y)=0$.