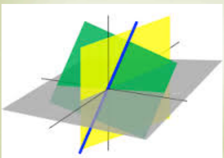
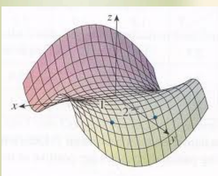
Double Integrals Over Rectangles, Iterated Integrals



$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

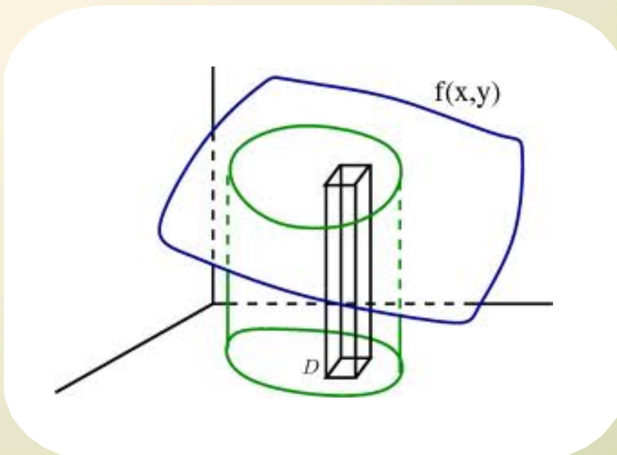
$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y \right]_{x=0}^{x=2y} dy$$

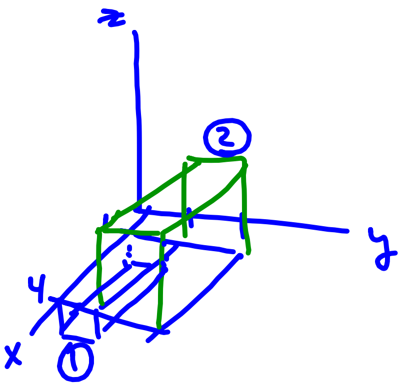
$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2} \right]_{y=0}^{y=1} = \frac{1}{2}$$

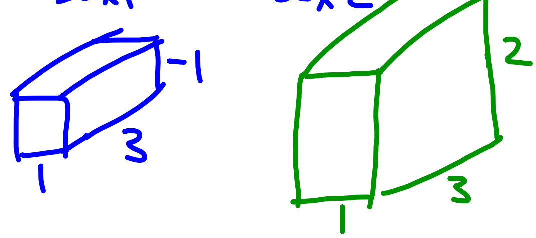


EX 1 If $f(x,y)=\begin{cases} -1 & 1\le x\le 4,\ 0\le y<1 \\ 2 & 1\le x\le 4,\ 1\le y\le 2 \end{cases}$

find the signed volume between the $z = f(x,y)$ surface and the xy-plane.



$$V = V_{box1} + V_{box2}$$



$$V = 1(3)(-1) + 1(3)(2) = 3$$

Definition (Double Integral)

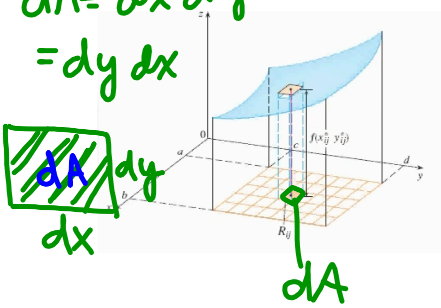
Let $z = f(x,y)$ be defined on a closed rectangle, R.

If $\lim\limits_{|p|\to 0} \sum\limits_{k=1}^{n} f(\bar{x}_k, \bar{y}_k)\Delta A_k$ exists, then f is integrable over R

and the double integral $\underbrace{\iint\limits_R f(x,y)\,dA}_{\text{total volume over } R} = \lim\limits_{|p|\to 0} \sum\limits_{k=1}^{n} f(\bar{x}_k, \bar{y}_k)\Delta A_k$.

Note: $\int_a^b f(x)\,dx$ = area under $y = f(x)$ curve

$dA = dx\,dy$
$= dy\,dx$



$\underbrace{f(\bar{x}_k, \bar{y}_k)}_{\text{height of each box}}\ \underbrace{\Delta A_k}_{\text{a little bit of area (base of box)}}$

Integrability Theorem

If f is continuous on the closed rectangle R, then f is integrable on R.

$f = f(x,y)$ (R is in xy-plane)

(ie.. I can find the signed volume of $z = f(x,y)$ over R)

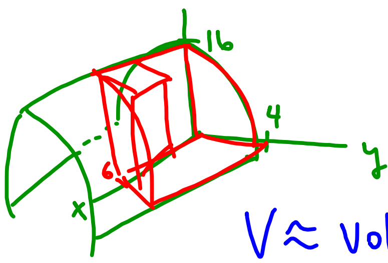
EX 2 Let $R = \{(x,y) \mid 0 \le x \le 6,\quad 0 \le y \le 4\}$ and $f(x,y) = 16 - y^2$.

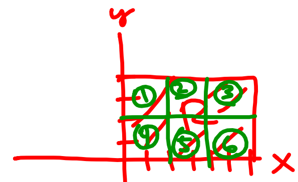
Partition R into 6 equal squares by lines $x = 2$, $x = 4$ and $y = 2$.

Approximate $\iint\limits_R f(x,y)\,dA$ as $\sum\limits_{k=1}^{6} f(\bar{x}_k, \bar{y}_k)\Delta A_k$

where $(\bar{x}_k, \bar{y}_k)$ are centers of squares.

$z = 16 - y^2$ (cylinder along x axis)



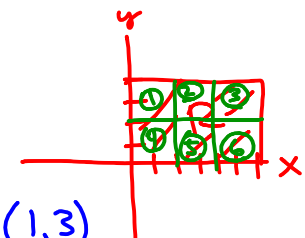


(base of rect. boxes)

$V \approx$ volume of 6 rect. boxes

base of each box $\square$ 2, 2 $\Rightarrow A_{base} = 4$

ht of each box:



① center: $(1,3)$

ht of box over square ① $= f(1,3)$

② center: $(3,3)$

ht of box $= f(3,3)$

$$V \approx \underbrace{f(1,3)(4)}_{\text{box ①}} + \underbrace{f(3,3)(4)}_{\text{vol box ②}} + \underbrace{f(5,3)(4)}_{\text{box ③}}$$

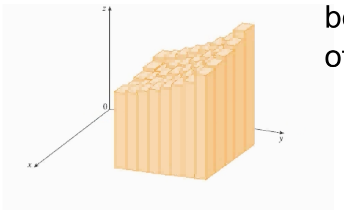
$$+ \underbrace{f(1,1)(4)}_{\text{box ④}} + \underbrace{f(3,1)(4)}_{\text{box ⑤}} + \underbrace{f(5,1)(4)}_{\text{box ⑥}}$$

$z = 16 - y^2$

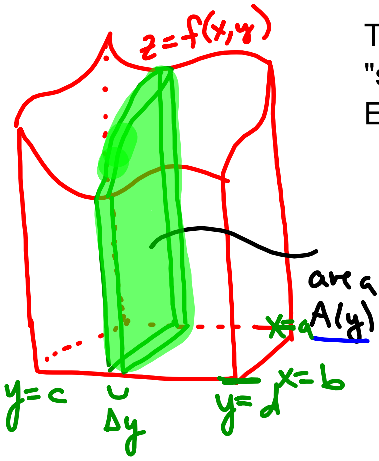
$$V \approx 4\Big[(16-3^2) + (16-3^2) + (16-3^2) + (16-1^2) + (16-1^2) + (16-1^2)\Big]$$

$$= 4(21 + 45) = 4(66) = \boxed{264} \text{ units}^3$$

Iterated Integrals



The total volume is the sum of many rectangular boxes and then we take the limit as the number of boxes goes to infinity to get the exact volume.



To find this volume, we can take thin "slab" cross-sections and add them up. Each slab has volume $A(y)\Delta y$.

$$\boxed{V=\int_c^d A(y)\,dy}$$ volume of slab

$$\boxed{A(y)=\int_a^b f(x,y)\,dx}$$ x goes from a to b (y is presumed to be fixed)

("a definite integral adds up a bunch of little somethings")

$$\Rightarrow V=\int_c^d\left(\int_a^b f(x,y)\,dx\right)dy$$

$$V=\int_c^d\int_a^b f(x,y)\,dx\,dy$$ (work from "inside out")

or

$$V=\int_a^b\int_c^d f(x,y)\,dy\,dx$$ (we can switch order of integrals easily here because we have fixed values of x and y, i.e. a, b, c, d are fixed.)

Properties of the Double Integral

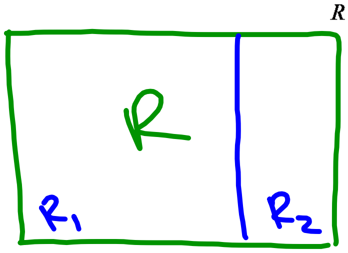
1) commutes w/ scalar multiplication

A) It is a linear operator 1) $\iint_R kf(x,y)\,dA = k\iint_R f(x,y)\,dA$

and 2) $\iint_R [f(x,y)+g(x,y)]\,dA = \iint_R f(x,y)\,dA + \iint_R g(x,y)\,dA$

2) distributes through addition.

B) Additive on rectangles $\iint_R f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$



Where R_1 and R_2 overlap only on a line segment and comprise all of all R.

$R = R_1 \cup R_2$

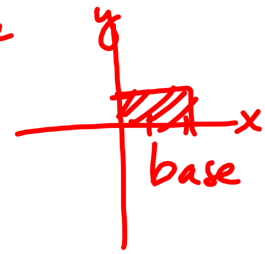
C) If $f(x,y) \le g(x,y)$, then $\iint_R f(x,y)\,dA \le \iint_R g(x,y)\,dA$

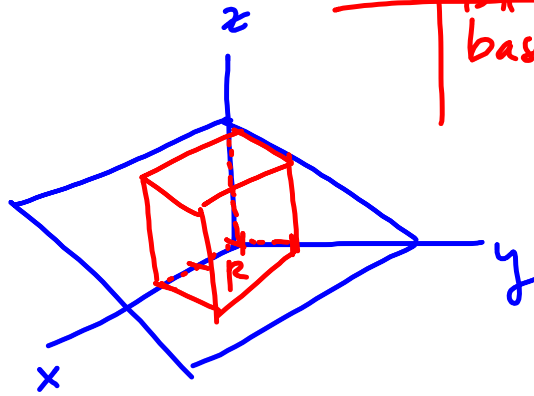
D) $\iint_R k\,dA = k\iint_R dA = kA(R)$ (k constant)

$z = k$

EX 3 Calculate $\iint_R f(x,y)\,dA$ where $f(x,y) = 7 - y$ plane

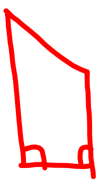
$R = \{(x,y) \mid 0 \le x \le 2, \quad 0 \le y \le 1\}$.





Hint: Sketch it and see if you recognize it.

side view:



right trapezoid

solid: trapezoidal prism

$$V = \iint_R f(x,y)\,dA = \int_0^1 \int_0^2 (7-y)\,dx\,dy$$

$$= \int_0^1 (7-y)\left(\int_0^2 dx\right) dy$$

$$= \int_0^1 (7-y)\left(x\Big|_0^2\right) dy$$

$$= \int_0^1 (7-y)(2)\,dy$$

$$= 2\left(7y - \frac{y^2}{2}\right)\Big|_0^1$$

$$= 2\left[\left(7-\tfrac{1}{2}\right) - 0\right] = 13 \text{ units}^3$$

Let's practice computing some double integrals.

EX 4 Evaluate $\int_0^4 \left[\int_{-1}^2 (x^2 - 3y)dx \right] dy$

$$= \int_0^4 \left(\frac{x^3}{3} - 3yx \right) \Big|_{-1}^{2} dy$$

$$= \int_0^4 \left(\left(\frac{8}{3} - 3y(2) \right) - \left(\frac{-1}{3} - 3y(-1) \right) \right) dy$$

$$= \int_0^4 (3 - 9y)\, dy = \left(3y - \frac{9y^2}{2} \right) \Big|_0^4 = 12 - 9(8) - 0$$

$$= -60$$

EX 5 $\int_0^1 \int_0^1 \frac{y}{(xy+1)^2} dx\, dy$

$$= \int_0^1 y \left(\int_0^1 \frac{1}{(xy+1)^2} dx \right) dy$$

$u = xy + 1$

$du = y\, dx$

$x = 0,\ u = 1$

$x = 1,\ u = y + 1$

$$= \int_0^1 y \left[\int_1^{y+1} \frac{1}{u^2} \left(\frac{du}{y} \right) \right] dy$$

$$= \int_0^1 \cancel{\frac{y}{y}} \left[\int_1^{y+1} u^{-2}\, du \right] dy$$

$$= \int_0^1 \left(\frac{u^{-1}}{-1} \Big|_1^{y+1} \right) dy$$

$$= \int_0^1 \left(\frac{-1}{y+1} - \frac{-1}{1} \right) dy = \int_0^1 \left(\frac{-1}{y+1} + 1 \right) dy$$

$$= \left[-\ln|y+1| + y \right] \Big|_0^1$$

$$= (-\ln 2 + 1) - (-\underbrace{\ln 1}_{0} + 0)$$

$$= \boxed{1 - \ln 2}$$

EX 6 $\iint_R xy\sqrt{1+x^2}\,dA$ $\qquad R=\{(x,y)\,|\,0\le x\le\sqrt{3}, 1\le y\le 2\}$

$$= \int_0^{\sqrt{3}}\int_1^2 xy\sqrt{1+x^2}\,dy\,dx$$

$$=\left(\int_0^{\sqrt{3}} x\sqrt{1+x^2}\,dx\right)\cdot\left(\int_1^2 y\,dy\right)$$

$$= \int_0^{\sqrt{3}} x\sqrt{1+x^2}\left(\int_1^2 y\,dy\right)dx$$

$$= \int_0^{\sqrt{3}} x\sqrt{1+x^2}\left(\frac{y^2}{2}\Big|_1^2\right)dx$$

$$= \int_0^{\sqrt{3}} x\sqrt{1+x^2}\left(2-\tfrac{1}{2}\right)dx$$

$$= \frac{3}{2}\int_0^{\sqrt{3}} x\sqrt{1+x^2}\,dx$$

$u=1+x^2$

$du=2x\,dx$

$\frac{1}{2}du=x\,dx$

$x=0,\ u=1+0^2=1$

$x=\sqrt{3},\ u=1+(\sqrt{3})^2=4$

$$= \frac{3}{2}\int_1^4 \frac{1}{2}\sqrt{u}\,du$$

$$= \frac{3}{4}\int_1^4 u^{1/2}\,du = \frac{3}{4}\left(u^{3/2}\left(\frac{2}{3}\right)\right)\Big|_1^4$$

$$= \frac{1}{2}\left(4^{3/2}-1^{3/2}\right)$$

$$= \frac{1}{2}(8-1) = \boxed{7/2}$$

Note: If we have $f(x,y)=g(x)h(y)$

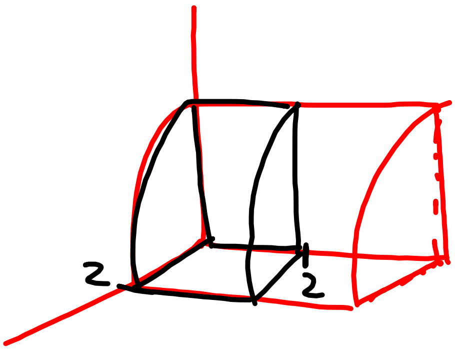
then $\int_a^b\int_c^d f(x,y)\,dy\,dx$

$$= \int_a^b\int_c^d g(x)\,\underline{h(y)\,dy}\,dx$$

$$= \left[\int_a^b g(x)\,dx\right]\left[\int_c^d h(y)\,dy\right]$$

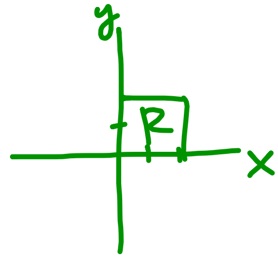
most of the time $f(x,y)\ne h(y)g(x)$

EX 7 Find the volume of the solid in the first octant enclosed by $z = 4\text{-}x^2$ and $y = 2$.



$$V = \iint_R (4-x^2)\, dA$$

domain:
region



$$\Rightarrow V = \int_0^2 \int_0^2 (4-x^2)\, dx\, dy$$

$$= \left(\int_0^2 dy\right)\left(\int_0^2 (4-x^2)\, dx\right)$$

$$= \left(y\Big|_0^2\right)\left(\left(4x-\frac{x^3}{3}\right)\Big|_0^2\right)$$

$$= (2-0)\left(\left(8-\frac{8}{3}\right)-0\right)$$

$$= 2\left(\frac{16}{3}\right) = \boxed{\frac{32}{3}}$$