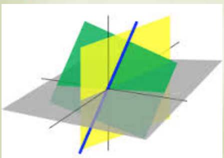
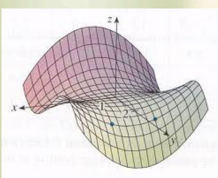


$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$

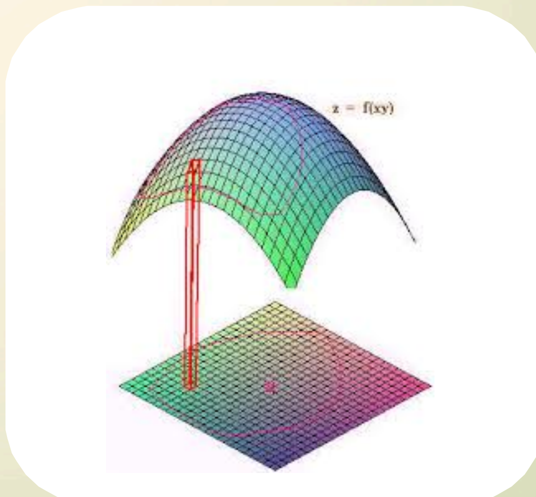


$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$

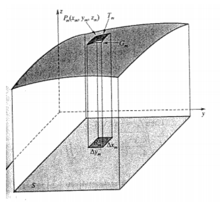
$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

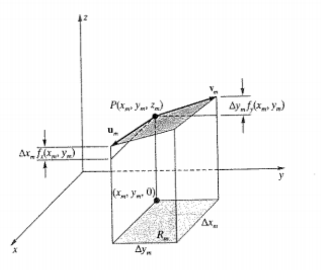
$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Surface Area



Surface Area





To find the surface area, we are going to add up lots of little areas of parallelograms that are tangent to the surface.

In the limit as Δx and Δy go to zero, the sum becomes an integral which gives the true surface area.

vectors that are sides of the m^{th} parallelogram

$$\vec{u}_m = \Delta x_m \hat{i} + f_x(x_m, y_m)\Delta x_m \hat{k} = \langle dx_m, 0, f_x(x_m, y_m)dx_m \rangle$$

$$\vec{v}_m = \Delta y_m \hat{j} + f_y(x_m, y_m)\Delta y_m \hat{k} = \langle 0, dy_m, f_y(x_m, y_m)dy_m \rangle$$

We know that the area of the parallelogram is the length of the cross product of its vector sides.

$A(T_m)$ = area of m^{th} tangent parallelogram

$$= \|\vec{u}_m \times \vec{v}_m\|$$

$$\vec{u}_m \times \vec{v}_m = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \Delta x_m & 0 & f_x \Delta x_m \\ 0 & \Delta y_m & f_y \Delta y_m \end{vmatrix}$$

$$\vec{u}_m \times \vec{v}_m = \Delta x_m \Delta y_m \left(-f_x(x_m, y_m)\hat{i} - f_y(x_m, y_m)\hat{j} + \hat{k}\right)$$

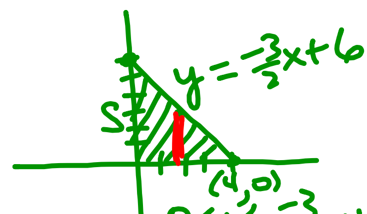
$$\Rightarrow A(T_m) = \underbrace{\Delta x_m \Delta y_m}_{\text{area of } m^{th} \text{ rectangle} = dA_m \text{ (in domain space)}} \sqrt{(f_x(x_m, y_m))^2 + (f_y(x_m, y_m))^2 + 1}$$

$$\Rightarrow SA = \iint_S \text{area of tangent parallelograms}$$

$$\boxed{SA = \iint_S \sqrt{f_x^2 + f_y^2 + 1}\; dA}$$

EX 1 Find the surface area of the plane $\overbrace{3x - 2y + 6z = 12}^{f}$ that is bounded by the planes, $\underbrace{x = 0,\ y = 0, \text{ and } 3x + 2y = 12}_{S}$.

domain space



$0 \le y \le -\frac{3}{2}x + 6$

$0 \le x \le 4$

$f_x = -\frac{1}{2}$

$f_y = \frac{1}{3}$

$$6z = 12 - 3x + 2y$$

$$z = f(x,y) = 2 - \frac{1}{2}x + \frac{1}{3}y$$

$$SA = \iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

$$= \int_0^4 \int_0^{-\frac{3}{2}x+6} \sqrt{\frac{1}{4} + \frac{1}{9} + 1}\, dy\, dx$$

$$= \int_0^4 \frac{7}{6} y \Big|_0^{-\frac{3}{2}x+6} dx$$

$$= \frac{7}{6} \int_0^4 \left(-\frac{3}{2}x + 6\right) dx$$

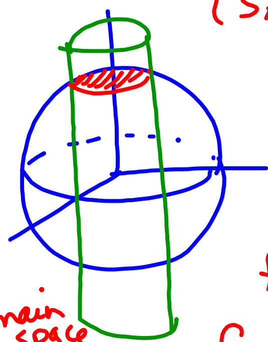
$$= \frac{7}{6} \left(-\frac{3}{4}x^2 + 6x\right)\Big|_0^4$$

$$= \frac{7}{6} \left(-\frac{3}{4}(4^2) + 6(4) - 0\right)$$

$$= \frac{7}{6}(12) = \boxed{14} \text{ units}^2$$

EX 2 Find the surface area for the part of the sphere, $\overbrace{x^2 + y^2 + z^2 = 9}^{f}$, that is inside the circular cylinder, $\underbrace{x^2 + y^2 = 4}_{S}$.

(SA of top piece)

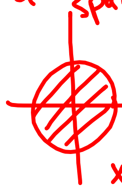


$$SA = \iint_S \sqrt{f_x^2 + f_y^2 + 1}\, dA$$

$$f(x,y) = z = \sqrt{9 - x^2 - y^2}$$

$$f_x = \frac{-x}{\sqrt{9-x^2-y^2}}, \quad f_y = \frac{-y}{\sqrt{9-x^2-y^2}}$$

domain space



S

$x^2+y^2=4$

$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2}$

$-2 \le x \le 2$

$0 \le r \le 2$

$0 \le \theta \le 2\pi$

Switch to polar coords

$$f_x = \frac{-r\cos\theta}{\sqrt{9-r^2}}, \quad f_y = \frac{-r\sin\theta}{\sqrt{9-r^2}}$$

$$\Rightarrow \sqrt{f_x^2+f_y^2+1} = \sqrt{\frac{r^2\cos^2\theta + r^2\sin^2\theta}{9-r^2} + 1}$$

$$= \sqrt{\frac{r^2 + 9 - r^2}{9-r^2}} = \frac{3}{\sqrt{9-r^2}}$$

$$\Rightarrow SA = \int_0^{2\pi}\int_0^2 \frac{3}{\sqrt{9-r^2}}\, r\, dr\, d\theta$$

$u = 9 - r^2$

$du = -2r\, dr$

$-\frac{1}{2} du = r\, dr$

$r = 0,\ u = 9 - 0 = 9$

$r = 2,\ u = 9 - 4 = 5$

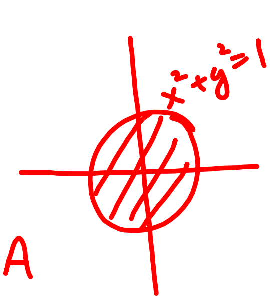
$$= \int_0^{2\pi}\int_9^5 \frac{-3}{2} u^{-1/2}\, du\, d\theta$$

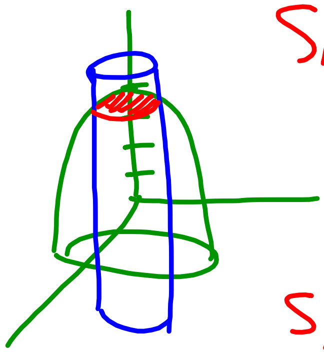
$$= \int_0^{2\pi} -3u^{1/2}\Big|_9^5\, d\theta$$

$$= (2\pi)\left(-3(\sqrt{5} - \sqrt{9})\right)$$

$$= \boxed{6\pi(3 - \sqrt{5})} \text{ units}^2$$

EX 3 Find the surface area of $z = \overbrace{4 - x^2 - y^2}^{f(x,y)}$
over $S = \{(x,y) \mid x^2 + y^2 \le 1\}$.





$$SA = \iint_S \sqrt{f_x^2 + f_y^2 + 1}\ dA$$

$0 \le r \le 1$

$0 \le \theta \le 2\pi$

$$f_x = -2x, \quad f_y = -2y$$

$$SA = \int_0^{2\pi}\int_0^1 \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + 1}\ r\,dr\,d\theta$$

$$SA = \int_0^{2\pi}\int_0^1 \sqrt{4r^2+1}\ r\,dr\,d\theta$$

$u = 4r^2 + 1$

$du = 8r\,dr$

$\frac{1}{8}du = r\,dr$

$r = 0,\ u = 1$

$r = 1,\ u = 4 + 1 = 5$

$$= \int_0^{2\pi}\int_1^5 \frac{\sqrt{u}}{8}\,du\,d\theta$$

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_1^5 \frac{1}{8}u^{1/2}\,du\right)$$

$$= 2\pi\left(\frac{1}{8}\right)\frac{2}{3}u^{3/2}\Big|_1^5$$

$$= \frac{\pi}{4}\left(5^{3/2} - 1\right) \text{ units}^2$$

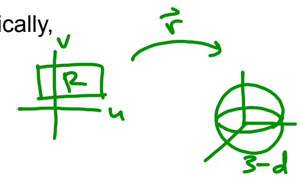
Note: remember from Calc 1.

Length of curve

$$L = \int_a^b \sqrt{\left(\frac{df}{dx}\right)^2 + 1}\ dx$$

For a surface area defined parametrically,

$\vec{r}(u,v) = \langle f(u,v),\ g(u,v),\ h(u,v) \rangle$.



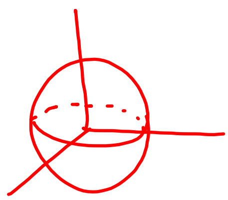
$$SA = \iint_R \|\vec{r}_u \times \vec{r}_v\| \, dA$$
$$= \iint_R \|\vec{r}_u \times \vec{r}_v\| \, du\, dv$$

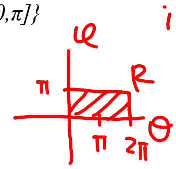
EX 4 Find the surface area of a surface given parametrically by

$\vec{r}(\theta,\varphi) = \langle 2\sin\varphi\cos\theta,\ 2\sin\varphi\sin\theta, 2\cos\varphi \rangle$.

$R = \{(\theta,\varphi) \mid \theta \notin [0,2\pi],\ \varphi \in [0,\pi]\}$

("uv space" is $\theta\varphi$ space)





$$\vec{r}_\theta = \langle -2\sin\varphi\sin\theta,\ 2\sin\varphi\cos\theta,\ 0 \rangle$$
$$\vec{r}_\varphi = \langle 2\cos\varphi\cos\theta,\ 2\cos\varphi\sin\theta,\ -2\sin\varphi \rangle$$

$$\vec{r}_\theta \times \vec{r}_\varphi = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -2\sin\varphi\sin\theta & 2\sin\varphi\cos\theta & 0 \\ 2\cos\varphi\cos\theta & 2\cos\varphi\sin\theta & -2\sin\varphi \end{vmatrix}$$

$$= -4\sin^2\varphi\cos\theta\,\hat{i} - 4\sin^2\varphi\sin\theta\,\hat{j} + (-4\sin\varphi\cos\varphi\sin^2\theta - 4\sin\varphi\cos\varphi\cos^2\theta)\hat{k}$$

$$= -4\sin^2\varphi\cos\theta\,\hat{i} - 4\sin^2\varphi\sin\theta\,\hat{j} - 4\sin\varphi\cos\varphi\,\hat{k}$$

$$= -4\sin\varphi\left[\sin\varphi\cos\theta\,\hat{i} + \sin\varphi\sin\theta\,\hat{j} + \cos\varphi\,\hat{k}\right]$$

$$\|\vec{r}_\theta \times \vec{r}_\varphi\| = \sqrt{16\sin^2\varphi\,\underbrace{\left[\sin^2\varphi\cos^2\theta + \sin^2\varphi\sin^2\theta + \cos^2\varphi\right]}_{1}}$$

$$= 4\sin\varphi$$

$$A = \int_0^\pi \int_0^{2\pi} 4\sin\varphi \quad d\theta\, d\varphi$$
$$= \int_0^\pi 4(2\pi)\sin\varphi \, d\varphi$$
$$= -8\pi\cos\varphi \Big|_0^\pi = -8\pi(-1-1)$$
$$= \boxed{16\pi}$$