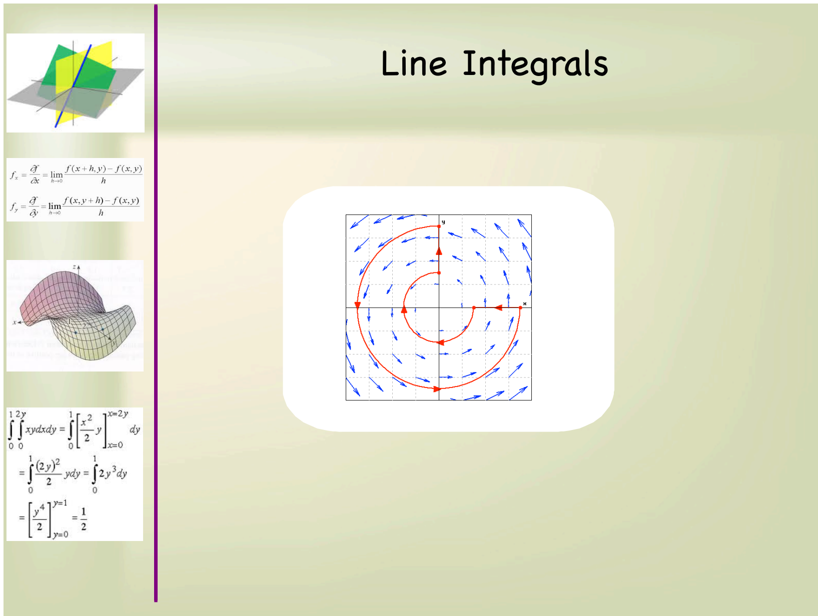


Let's review parameterization of curves.

The length of a parameterized curve in 2-D $(x(t),y(t)),\ t \in [a,b]$ is given by

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt\ .$$

In 3-D if $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$,

then the length of a curve is

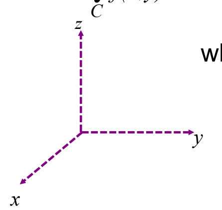
$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Suppose $f(x,y)$ is a function whose domain contains the curve

$C: \vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}, \quad t \in [a,b]$.

The line integral of f along the curve C from a to b is defined

as $\int_C f(x,y)ds$



where ds = arc length differential.

We know that $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

Line integral = $\int_a^b f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_C f(x,y)ds$

In 3 variables

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\,|\vec{v}(t)|\,dt$$

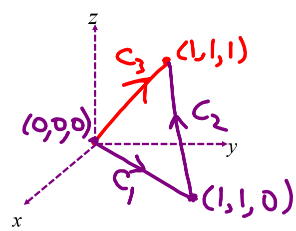
$$= \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

where $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath} + z(t)\hat{k}$

$\vec{v}(t) = x'(t)\hat{\imath} + y'(t)\hat{\jmath} + z'(t)\hat{k}$

EX 1 The figure shows two different paths, $C_1 \cup C_2$ and C_3.

Find $\displaystyle\int_{C_3} (x-3y^2+z)\,ds$ and $\displaystyle\int_{C_1 \cup C_2} (x-3y^2+z)\,ds$.



EX 2 A thin wire is bent in the shape of the semicircle

$x = a\cos t, \quad t \in [0,\pi], \quad a > 0$

$y = a\sin t$

If the density of the wire is proportional to the distance from the x-axis, find the mass of the wire.

Work

The goal is to calculate the work done by a vector field $\vec{F}(x,y,z)$ in moving an object along a curve C with parameterization.

$C: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $t \in [a,b]$

The work done to move the object at (x,y,z) by a small vector, $\Delta \vec{r}$ is

$$\Delta W = \vec{F}(x,y,z) \cdot \Delta \vec{r}(x,y,z)$$

$$W = \int_C \vec{F} \cdot d\vec{r}$$

Formula for calculating work

If $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$

where $M = M(x,y,z)$
$N = N(x,y,z)$
$P = P(x,y,z)$

$$\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$

$$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

then $W = \int_C \vec{F} \cdot d\vec{r} =$

EX 3 Find the work done by an inverse square law force field

$$\vec{F}(x,y,z) = \frac{-c(x\hat{i} + y\hat{j} + z\hat{k})}{\sqrt{x^2 + y^2 + z^2}}$$

in moving a particle along the straight line curve from $(0,3,0)$ to $(4,3,0)$.

Note: If $c > 0$, then the work done is negative.

EX 4 Evaluate $\int_C (2x + 9z)ds$, where C is the curve given by

$x = t,\ y = t^2, z = t^3,\ \ t \in [0,1]$.

EX 5 Evaluate $\int_C \left(ydx + x^2 dy \right)$, where C is the curve given by

$x = 2t,\ y = t^2 - 1,\ t \in [0,2]$.