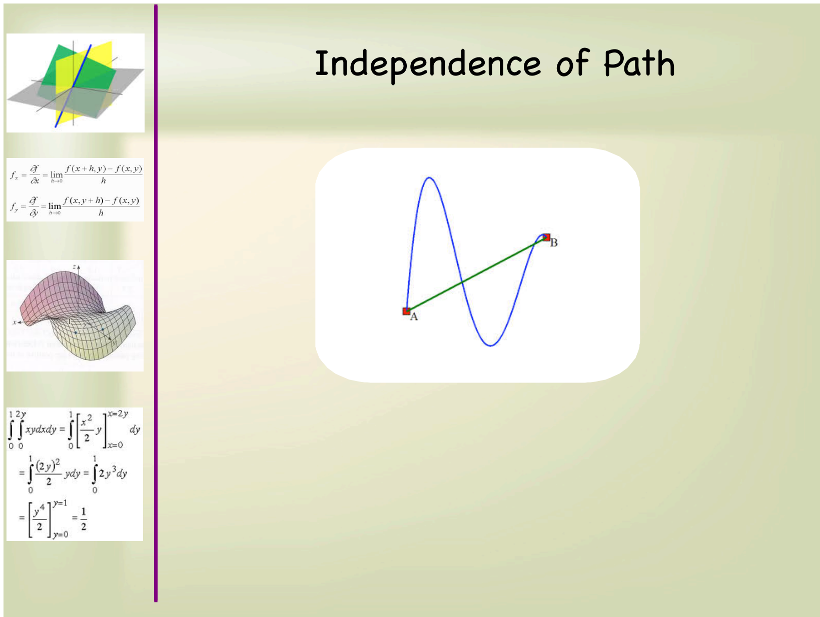


Recall the Fundamental Theorem of Calculus.

$$\int_a^b f'(x)dx = f(b) - f(a)$$

We would like an analogous theorem for line integrals.

<u>Fundamental Theorem of Line Integrals</u>

Let $C$ be the curve given by the parameterization $\vec{r}(t),\ t \in [a,b]$, such that $\vec{r}(t)$ is differentiable. If $f(\vec{r})$ is continuously differentiable on an open set containing $C$,

then

$$\int_C \nabla f(\vec{r}) \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

EX 1 Find work done by $\nabla f$ along a curve going from

(1,1,1) to (4,-1,2), given $f(r) = \frac{c}{\|\vec{r}\|} \nabla f = \frac{-c\vec{r}}{\|\vec{r}\|}$ .

A set, $D$, is called a <u>Path-Connected Set</u> if any 2 points in $D$ can be joined by a piece-wise smooth curve lying entirely in $D$.

Example   Non-example

What does it mean to be independent of path?

<u>Independence of Path Theorem</u>

Let $\vec{F}(\vec{r})$ be continuous on an open connected set $D$.

Then $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ is independent of any path, $C$, in $D$ iff $\vec{F}(\vec{r}) = \nabla f(\vec{r})$ for some $f(\vec{r})$ (scalar function),
i.e. if $\vec{F}(\vec{r})$ is a conservative vector field on $D$.

Equivalent Conditions for Line Integrals

Let $\vec{F}(\vec{r})$ be continuous on an open connected set $D$.
The following statements are equivalent.

a) $\vec{F} = \nabla f$ for some $f$ (i.e. $\vec{F}$ is conservative on $D$).

b) $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ is independent of the path, $C$, in $D$.

c) $\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = 0$ for every closed path in $D$.

Theorem

Let $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ with $M, N, P$ continuously differentiable on a ball, $D$.

Then $\vec{F}$ is conservative $\Leftrightarrow$ $\nabla \times \vec{F} = \vec{0}$.

Note:

If $\vec{F} = M\hat{i} + N\hat{j}$

then
$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ M & N & 0 \end{vmatrix} = \hat{k}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

and
$$\nabla \times \vec{F} = \vec{0} \Rightarrow \frac{dN}{dx} = \frac{dM}{dy}$$

EX 2 Is $\vec{F} = (12x^2 + 3y^2 + 5y)\,\hat{i} + (6xy - 3y^2 + 5x)\,\hat{j}$ conservative?

EX 3 Using $\vec{F}$ from Example 1, find $f$ such that $\vec{F} = \nabla f$.

EX 4 Using $\vec{F} = (12x^2 + 3y^2 + 5y)\,\hat{i} + (6xy - 3y^2 + 5x)\,\hat{j}$

calculate $\int_C \vec{F}(\vec{r}) \cdot d\vec{r}$ where $C$ is any path from $(0,0)$ to $(2,1)$.

EX 5 Show that the line integral $\int_C ((yz+1)dx + (xz+1)dy + (xy+1)dz)$

is independent of path and evaluate the integral, where $C$ is a curve from (0,1,0) to (1,1,1).

EX 6 Let $\vec{F} = (1 + 2xy\sin(x^2y))\hat{i} + (1 + x^2\sin(x^2y))\hat{j}$

Is $\vec{F}$ conservative?
If yes, then find $f$ such that $\vec{F} = \nabla f$.

EX 7 Evaluate $\int_{(0,0)}^{(1,\pi/2)} \left(e^x \sin y\,dx + e^x \cos y\,dy\right)$.