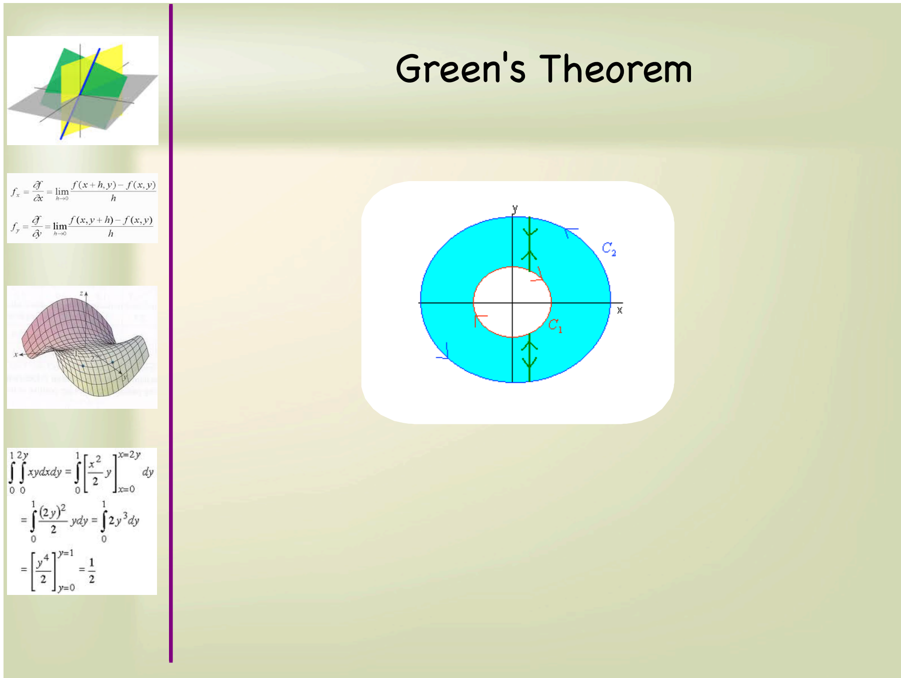

Goal:

Describe the relation between the way a fluid flows along or across the boundary of a plane region and the way fluid moves around inside the region.

Circulation or flow integral

Assume $\vec{F}(x,y)$ is the velocity vector field of a fluid flow. At each point (x,y) on the plane, $\vec{F}(x,y)$ is a vector that tells how fast and in what direction the fluid is moving at the point (x,y).

Assume $\vec{r}(t)=x(t)\hat{i} + y(t)\hat{j},\ t\in [a,b]$, is parameterization of a closed curve lying in the region of fluid flow.

Let $\vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$.

We want to measure "how much" fluid is moving along the curve $\vec{r}(t)$.

EX 1 Let $\vec{r}(t)$ be the parameterization of the unit circle centered at the origin. Draw these vector fields and think about how the fluid moves around that circle.

$\vec{F}(x,y) = -2\hat{i}$ $\qquad$ $\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2 + y^2}}$

When $\vec{F}(x,y)$ is parallel to the tangent line at a point, then the maximum flow is along a circle.

When $\vec{F}(x,y)$ is perpendicular to the tangent line at a point, then there is no flow along the circle.

So $\vec{F}(x,y)\cdot\vec{T}(x,y)$ measures the flow along the circle where $\vec{T}(x,y)=\vec{r}\,'(t)$.

We define the circulation of $\vec{F}$ along C, a parameterized curve from $t = a$ to $t = b$ as

$$\int_a^b \vec{F}(x,y)\cdot\vec{r}\,'(t)dt = \int_a^b \vec{F}\cdot d\vec{r} = \int_{t=a}^{t=b} Mdx + Ndy$$

EX 2 Given C: $\quad x = a\cos t, \quad t \in [0,2\pi]$

$y = a\sin t,$

find the circulation along C for each of these.

a) $\vec{F}_1(x,y) = 2\hat{i}$ $\qquad$ b) $\vec{F}_2(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2 + y^2}}$

Flux across a curve

Given $\vec{F}(x,y) = M\hat{i} + N\hat{j}$ (vector velocity field) and a curve C, with the parameterization $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$, $t \in [a,b]$, such that C is a positively oriented, simple, closed curve.

We want to know the rate at which a fluid is entering and leaving the area of the region enclosed by a curve, C. This is called flux.

$\vec{F}(x,y) \cdot \vec{n}(x,y)$ is the component of $\vec{F}$ perpendicular to the curve, so flux = $\oint_C \vec{F} \cdot \vec{n}\, ds$.

Now to find
$$\vec{n} = \vec{T} \times \hat{k}$$
$$= \left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} \right) \times \hat{k}$$
$$= \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j}$$

This means
$$\vec{F} \cdot \vec{n} = M\frac{dy}{ds} - N\frac{dx}{ds}$$
$$\text{flux} = \oint_C \left(M\frac{dy}{ds} - N\frac{dx}{ds} \right) ds$$
$$= \oint_C M dy - N dx$$

EX 3 Find the flux across C: $\vec{r}(t) = (a \cos t)\hat{i} + (a \sin t)\hat{j}$, $t \in [0,2\pi]$

a) $\vec{F}_1(x,y) = -2\hat{i}$

b) $\vec{F}_2(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2 + y^2}} = (-\sin t)\hat{i} + (\cos t)\hat{j}$

c) $\vec{F}_3(x,y) = x\hat{i} + y\hat{j} = (a \cos t)\hat{i} + (a \sin t)\hat{j}$

Two Forms of Green's Theorem
in The Plane

Let $\vec{F}(x,y) = M\hat{i} + N\hat{j}$

Let C be a simple, closed, positively oriented curve enclosing a region R in the xy-plane.

$$\oint_C M dy - N dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy$$

$$\oint_C \vec{F} \cdot \vec{n}\; ds = \iint_R \nabla \cdot \vec{F}\; dA$$

Let $\vec{F}(x,y) = M\hat{i} + N\hat{j}$

Let C be a simple, closed, positively oriented curve enclosing a region R in the xy-plane.

$$\oint_C M\, dx + N\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\; dy$$

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot \vec{T}\, ds = \iint_R \nabla \times \vec{F} \cdot \hat{k}\; dA$$

EX 5 Verify both forms of Green's theorem for the field

$\vec{F}(x,y) = (x\text{-}y)\hat{i} + x\hat{j}$

and the region R bounded by the circle

$C: \vec{r}(t) = (\cos t)\hat{i} + (\sin t)\hat{j},\ \ t \in [0,2\pi]$.

EX 6 Evaluate the integral $\oint_C (xy\,dy - y^2\,dx)$ where C is the square cut from the first quadrant by the lines $x = 1$ and $y = 1$.

EX 7 Calculate the flux of the field $\vec{F}(x,y) = x\hat{i} + y\hat{j}$ across the square bounded by the lines $x = \pm 1$ and $y = \pm 1$.