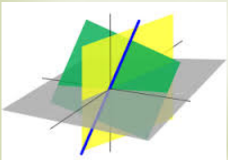
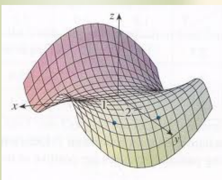


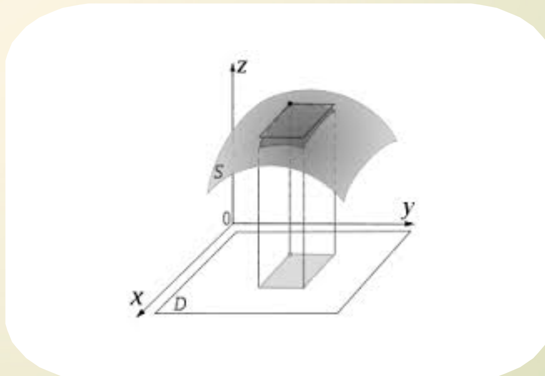
$$f_x = \frac{\partial f}{\partial x} = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$



$$\int_0^1\int_0^{2y} xy\,dx\,dy = \int_0^1\left[\frac{x^2}{2}y\right]_{x=0}^{x=2y}dy$$
$$= \int_0^1 \frac{(2y)^2}{2}y\,dy = \int_0^1 2y^3\,dy$$
$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

Surface Integrals



<u>Surface Integrals</u>

Let G be defined as some surface, $z = f(x,y)$.

The surface integral is defined as

$\iint_G g(x,y,z)\, dS$, where dS is a "little bit of surface area."

To evaluate we need this <u>Theorem</u>:

Let G be a surface given by $z = f(x,y)$ where (x,y) is in R, a bounded, closed region in the xy-plane.

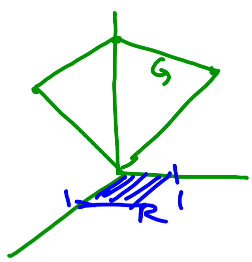
R is domain space in xy-plane

If f has continuous first-order partial derivatives and $g(x,y,z) = g(x,y,f(x,y))$ is continuous on R, then

surface integral

$$\boxed{\iint_G g(x,y,z)\,dS = \iint_R g(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\; dy\, dx}$$

EX 1 Evaluate $\iint_G g(x,y,z)\, dS$ given by $g(x,y,z) = x$, and G is the plane $\underbrace{x + y + 2z = 4}_{G}$, $\underbrace{x \in [0,1],\ y \in [0,1]}_{R \text{ (unit square)}}$.



$$\iint_G x\, dS$$

$$G: \ z = 2 - \tfrac{1}{2}x - \tfrac{1}{2}y$$

$$z = f(x,y)$$

$$dS = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$$

$$f_x = -\tfrac{1}{2},\ f_y = -\tfrac{1}{2}$$

$$dS = \sqrt{\tfrac{1}{4} + \tfrac{1}{4} + 1}\, dx\, dy$$

$$= \sqrt{\tfrac{3}{2}}\, dx\, dy$$

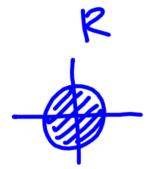
$$\iint_G g\, dS = \iint_R x\sqrt{\tfrac{3}{2}}\, dy\, dx = \int_0^1\int_0^1 \sqrt{\tfrac{3}{2}}\, x\, dy\, dx$$

$$= \sqrt{\tfrac{3}{2}} \int_0^1 x\left(y\Big|_0^1\right) dx$$

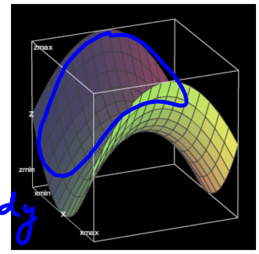
$$= \sqrt{\tfrac{3}{2}}\,(1) \int_0^1 x\, dx$$

$$= \sqrt{\tfrac{3}{2}}\left(\tfrac{1}{2}\right)$$

EX 2 Evaluate $\iint_G (2y^2+z)\, dS$ where G is the surface



$\underbrace{f(x,y) = z = x^2 - y^2}$, with R given by $\underbrace{0 \le x^2 + y^2 \le 1}_{R}$.



$$\iint_G (2y^2+z)\, dS$$

$$= \iint_R (2y^2+z)\sqrt{(2x)^2+(-2y)^2+1}\, dx\, dy$$

replace w/ $f(x,y)$

(switch to polar coords)

$$= \int_0^{2\pi}\int_0^1 \left(2(r\sin\theta)^2+(r\cos\theta)^2-(r\sin\theta)^2\right)\sqrt{4r^2+1}\; r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \left(r^3\sqrt{4r^2+1}\right) dr\, d\theta$$

$u = 4r^2+1 \iff r^2 = \frac{u-1}{4}$ | $r=0,\ u=1$

$du = 8r\, dr$ | $r=1,\ u=5$

$\frac{1}{8} du = r\, dr$

$$= \frac{1}{8}\int_0^{2\pi}\int_1^5 \left(\frac{u-1}{4}\right)\sqrt{u}\; du\, d\theta$$

$$= \frac{1}{32}\int_0^{2\pi}\int_1^5 \left(u^{3/2}-u^{1/2}\right) du\, d\theta$$

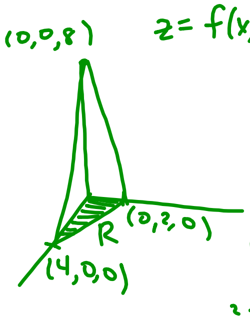
$$= \frac{1}{32}(2\pi)\left(\frac{2}{5}u^{5/2}-u^{3/2}\left(\frac{2}{3}\right)\right)\Big|_1^5$$

$$= \frac{\pi}{16}\left(\frac{2}{5}\left(5^{5/2}\right)-\frac{2}{3}\left(5^{3/2}\right)-\left(\frac{2}{5}-\frac{2}{3}\right)\right)$$

$$= \frac{\pi}{16}\left(10\sqrt{5}-\frac{10}{3}\sqrt{5}+\frac{4}{15}\right)$$

$$= \frac{\pi}{16}\left(\frac{20}{3}\sqrt{5}+\frac{4}{15}\right)$$

EX 3 Evaluate $\iint_G g(x,y,z)\, dS$ where $g(x,y,z) = z$ and G is the tetrahedron bounded by the coordinate planes and the plane $\underbrace{4x + 8y + 2z = 16}_{z = f(x,y)}$.



$z = 8 - 4y - 2x = f(x,y)$

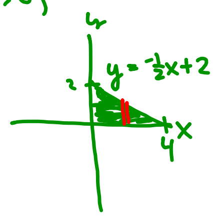
$f_x = -2, \quad f_y = -4$

$dS = \sqrt{4 + 16 + 1}\, dx\, dy$

$= \sqrt{21}\, dx\, dy$

$g(x,y,z) = z = 8 - 4y - 2x$

R:



$0 \le y \le -\frac{1}{2}x + 2$

$0 \le x \le 4$

$$\iint_S g\, dS = \iint_R g(x,y,f(x,y))\sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx$$

$$= \int_0^4 \int_0^{-\frac{1}{2}x+2} (8 - 4y - 2x)\sqrt{21}\, dy\, dx$$

$$= \sqrt{21}\int_0^4 \left(8y - 2y^2 - 2xy\right)\Big|_{y=0}^{y=-\frac{1}{2}x+2} dx$$

$$= \sqrt{21}\int_0^4 \left(8\left(-\tfrac{1}{2}x+2\right) - 2\left(-\tfrac{1}{2}x+2\right)^2 - 2x\left(-\tfrac{1}{2}x+2\right)\right) dx$$

$$= \sqrt{21}\int_0^4 \left(-4x + 16 - 2\left(\tfrac{1}{4}x^2 - 2x + 4\right) + x^2 - 4x\right) dx$$

$$= \sqrt{21}\int_0^4 \left(\tfrac{1}{2}x^2 - 4x + 8\right) dx$$

$$= \sqrt{21}\left(\tfrac{1}{6}x^3 - 2x^2 + 8x\right)\Big|_0^4$$

$$= \sqrt{21}\left(\left(\tfrac{64}{6} - 32 + 32\right) - 0\right)$$

$$= \boxed{\frac{32\sqrt{21}}{3}}$$

this answer is the surface integral integrated over "top" plane of tetrahedron, not through all other (3) faces!

Theorem

Let G be a smooth, two-sided surface given by $z = f(x,y)$, where (x,y) is in R and let $\vec{n}$ denote the upward unit normal on G. If f has continuous first-order partial derivatives and $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ is a continuous vector field, then the flux of $\vec{F}$ across G is given by

downward normal $\vec{n} = \langle f_x, f_y, -1 \rangle$

$$\text{flux } \vec{F} = \underbrace{\iint_G \vec{F}\cdot\vec{n}\, dS}_{} = \underbrace{\iint_R [-Mf_x - Nf_y + P]\,dx\,dy}_{} .$$

$f(x,y) = z$

$f(x,y) - z = 0$

notice similarity of this w/ flux across a curve formula! (which was $\int_C \vec{F}\cdot\vec{n}\, ds$)

need unit normal vector:

(outward) $\vec{n} = \langle -f_x, -f_y, 1 \rangle$

$$\hat{n} = \frac{\vec{n}}{\|\vec{n}\|} = \frac{\langle -f_x, -f_y, 1 \rangle}{\sqrt{f_x^2 + f_y^2 + 1}}$$

and $dS = \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy$

$\Rightarrow \vec{F}\cdot\vec{n}\, dS$

$$= \langle M, N, P \rangle \cdot \frac{\langle -f_x, -f_y, 1 \rangle}{\cancel{\sqrt{f_x^2 + f_y^2 + 1}}} \cancel{\sqrt{f_x^2 + f_y^2 + 1}}\, dx\, dy$$

$$= (-Mf_x - Nf_y + P)\, dx\, dy$$

EX 4 Evaluate the flux of $\vec{F}$ across G where

$\vec{F}(x,y,z) = (9-x^2)\hat{j}$ and G is the part of the plane

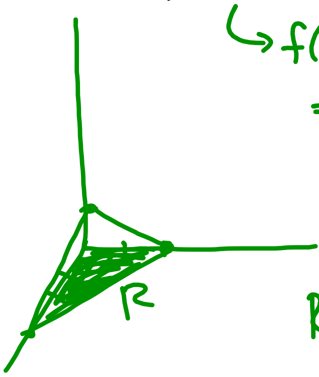
$2x + 3y + 6z = 6$ in the first octant.

$\hookrightarrow f(x,y) = z$

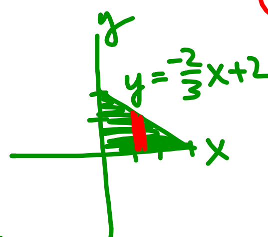
$= 1 - \frac{1}{2}y - \frac{1}{3}x$

flux $\vec{F}$

$= \iint_R \left(-Mf_x - Nf_y + P\right] dx\,dy$



R:



$0 \le y \le \frac{-2}{3}x + 2$

$0 \le x \le 3$

$$\text{flux } \vec{F} = \int_0^3 \int_0^{\frac{-2}{3}x+2} \left(0 - (9-x^2)\left(\frac{-1}{2}\right) + 0\right) dy\,dx$$

$$= \int_0^3 \int_0^{\frac{-2}{3}x+2} \left(\frac{9}{2} - \frac{1}{2}x^2\right) dy\,dx$$

$$= \int_0^3 \left(\frac{9}{2} - \frac{1}{2}x^2\right)\left(y \Big|_0^{\frac{-2}{3}x+2}\right) dx$$

$$= \int_0^3 \left(\frac{9}{2} - \frac{1}{2}x^2\right)\left(\frac{-2}{3}x + 2\right) dx$$

$$= \int_0^3 \left(-3x + 9 + \frac{1}{3}x^3 - x^2\right) dx$$

$$= \left(\frac{-3x^2}{2} + 9x + \frac{x^4}{12} - \frac{x^3}{3}\right)\Big|_0^3$$

$$= \left(\frac{-3}{2}(9) + 27 + \frac{27}{4} - 9\right) - 0$$

$$= 18 - \frac{27}{2} + \frac{27}{4} = \boxed{\frac{45}{4}}$$