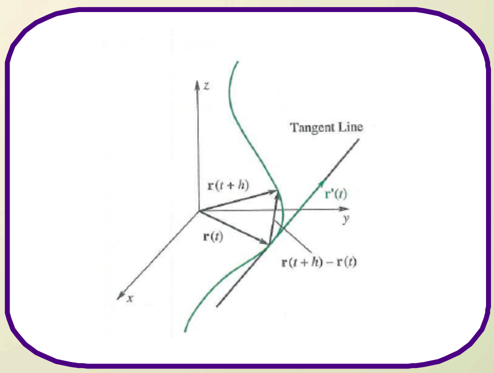
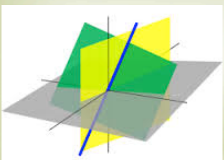
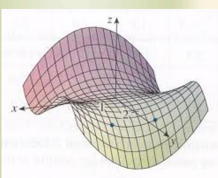
Lines and Tangent Lines in 3-Space





$$f_x = \frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h}$$

$$f_y = \frac{\partial f}{\partial y} = \lim_{h \to 0} \frac{f(x, y+h) - f(x, y)}{h}$$



$$\int_0^1 \int_0^{2y} xy\,dx\,dy = \int_0^1 \left[\frac{x^2}{2} y\right]_{x=0}^{x=2y} dy$$

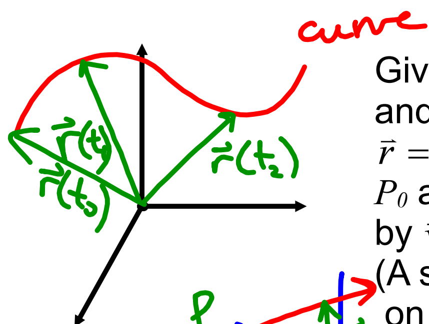
$$= \int_0^1 \frac{(2y)^2}{2} y\,dy = \int_0^1 2y^3\,dy$$

$$= \left[\frac{y^4}{2}\right]_{y=0}^{y=1} = \frac{1}{2}$$

A 3-D curve can be given parametrically by $x = f(t)$, $y = g(t)$ and $z = h(t)$ where t is on some interval I and f, g, and h are all continuous on I.

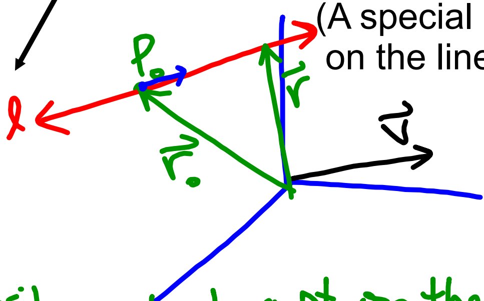
We could specify the curve by the position vector

$\vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$.



Given a point P_0, determined by the vector, $\vec{r}_0$ and a vector $\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$, the equation $\vec{r} = \vec{r}_0 + \vec{v}t$ determines a line passing through P_0 at $t = 0$ and heading in the direction determined by $\vec{v}$.

(A special case is when you are given two points on the line, P_0 and P_1, in which case $\vec{v} = \overrightarrow{P_0P_1}$.)



(positn vector to a pt on the line)

3-d line

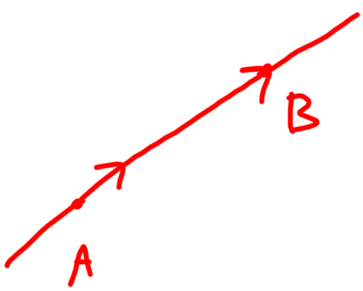
$\vec{r} = \langle x, y, z \rangle$, $\quad \vec{r}_0 = \langle x_0, y_0, z_0 \rangle \Rightarrow \boxed{\langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle a, b, c \rangle t}$

$$\boxed{x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct}$$

These become the <u>parametric equations</u> of a line in 3D where a,b,c are called <u>direction numbers</u> for the line (as are any multiples of a,b,c).

EX 1 Find parametric equations of a line through

A: *(2,-1,-5)* and B: *(7,-2,3)*.



directn of line:

$\vec{AB}$ (or $\vec{BA}$)

$\vec{AB} = \langle 7-2, -2-{-1}, 3-{-5} \rangle$

$\vec{v} = \langle 5, -1, 8 \rangle$

$P_0 = (2,-1,-5)$

line: $\langle x,y,z \rangle = \langle 2,-1,-5 \rangle + \langle 5,-1,8 \rangle t$

$$\boxed{\begin{cases} x = 2 + 5t \\ y = -1 - t \\ z = -5 + 8t \end{cases}}$$

Symmetric Equations for a line

$$x = x_0 + at,\ y = y_0 + bt,\ z = z_0 + ct$$

$a \neq 0$

$b \neq 0$

$c \neq 0$

(solve each of the above eqns for t)

$$\Rightarrow t = \frac{x - x_0}{a} \qquad t = \frac{y - y_0}{b} \qquad t = \frac{z - z_0}{c}$$

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

This is the line of intersection between the two planes given by

① $\dfrac{x - x_0}{a} = \dfrac{y - y_0}{b}$ and ② $\dfrac{y - y_0}{b} = \dfrac{z - z_0}{c}$.

EX 2 Write the symmetric equations for the line through $\overset{P_0}{(-2,2,-2)}$ and parallel to $\overset{\vec{v}}{\langle 7,-6,3\rangle}$.

parametric

$$\left.\begin{aligned} x &= -2+7t \\ y &= 2+-6t \\ z &= -2+3t \end{aligned}\right\}$$

symmetric

$$\boxed{\frac{x+2}{7}=\frac{y-2}{-6}=\frac{z+2}{3}}$$

EX 3 Find the symmetric equations of the line through $\overset{P_0}{(-5,7,-2)}$ and perpendicular to both $\overset{\vec{u}}{\langle 3,1,-3\rangle}$ and $\overset{\vec{w}}{\langle 5,4,-1\rangle}$.

$$\vec{v}=\vec{u}\times\vec{w}=\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & -3 \\ 5 & 4 & -1 \end{vmatrix} = \hat{i}\left(-1-(-12)\right)-\hat{j}\left(-3-(-15)\right)+\hat{k}\left(12-5\right)$$

$$= 11\hat{i}-12\hat{j}+7\hat{k}$$

$$\frac{x-(-5)}{11}=\frac{y-7}{-12}=\frac{z-(-2)}{7}$$

$$\boxed{\frac{x+5}{11}=\frac{y-7}{-12}=\frac{z+2}{7}}$$

EX 4 Find the symmetric equations of the line of intersection between the planes $\boxed{A}$ $x + y - z = 2$ and $\boxed{B}$ $3x - 2y + z = 3$.

① (technique from Int. Algebra)

(-3) $x+y-z=2$
$3x-2y+z=3$

use Gauss-Jordan elimination

$$\begin{cases} x+y-z=2 \\ -5y+4z=-3 \end{cases}$$

solve for z:

$4z = 5y - 3$

$$\boxed{z = \frac{5}{4}y - \frac{3}{4}}$$

plug into $\boxed{A}$:

$x + y - \left(\frac{5}{4}y - \frac{3}{4}\right) = 2$ solve for x.

$x - \frac{1}{4}y + \frac{3}{4} = 2$

$$\boxed{x = \frac{1}{4}y + \frac{5}{4}}$$

parametric eqns of line of intersection:

$$\begin{cases} x = \frac{1}{4}t + \frac{5}{4} \\ y = t \\ z = \frac{5}{4}t - \frac{3}{4} \end{cases}$$

② $\boxed{A}$ $x+y-z=2$ $\qquad \vec{n}_A = \langle 1,1,-1\rangle$

$\boxed{B}$ $3x-2y+z=3$ $\qquad \vec{n}_B = \langle 3,-2,1\rangle$

$$\vec{v} = \vec{n}_A \times \vec{n}_B = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ 3 & -2 & 1 \end{vmatrix} = \hat{\imath}(1-2) - \hat{\jmath}(1+3) + \hat{k}(-2-3)$$

$$= \langle -1,-4,-5\rangle$$

choose $\vec{v} = \langle 1,4,5\rangle$

need a pt: (from work in ①)

know $\left.\begin{aligned} z &= \tfrac{5}{4}y - \tfrac{3}{4} \\ x &= \tfrac{1}{4}y + \tfrac{5}{4} \end{aligned}\right\}$ choose $y=4$

$\Rightarrow z = 5 - \frac{3}{4} = \frac{17}{4}$

$x = 1 + \frac{5}{4} = \frac{9}{4}$

$P_0\left(\frac{9}{4}, 4, \frac{17}{4}\right)$

symmetric eqns of line:

$$\frac{x-9/4}{1} = \frac{y-4}{4} = \frac{z-17/4}{5}$$

$$\boxed{4x-9 = y-4 = \frac{4z-17}{5}}$$

Tangent Line to a Curve

If $\vec{r} = \vec{r}(t) = f(t)\hat{i} + g(t)\hat{j} + h(t)\hat{k}$ is a position vector along a curve in 3D,

then $\vec{r}\,'(t) = \lim_{h \to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h} \Rightarrow \vec{r}\,'(t) = f\,'(t)\hat{i} + g\,'(t)\hat{j} + h\,'(t)\hat{k}$

is a vector in the direction of the tangent line to the 3D curve. (This holds in 2D as well.)

EX 5 Find the parametric equations of the tangent line to the curve

$\underbrace{x = 2t^2,\ y = 4t,\ z = t^3}_{\text{3-d parametric curve}}$ at $t = 1$.

$\vec{r}(t) = 2t^2\hat{i} + 4t\hat{j} + t^3\hat{k}$

tangent vector: $\vec{r}\,'(t) = 4t\hat{i} + 4\hat{j} + 3t^2\hat{k}$

$\vec{v}$ (in direction of the tangent line)

$= \vec{r}\,'(1) = 4\hat{i} + 4\hat{j} + 3\hat{k} = \langle 4, 4, 3 \rangle$

Point on curve: $\vec{r}(1) = 2\hat{i} + 4\hat{j} + 1\hat{k}$

line: $\begin{cases} x = 2 + 4t \\ y = 4 + 4t \\ z = 1 + 3t \end{cases}$