The Game of SET®, and its Mathematics.

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But, as for everything else, so for a mathematical theory–beauty can be perceived but not explained.

   — A. Cayley

Introduction

The game of SET®was created by Marsha Jean Falco in 1974, as a positive side-effect of her research in population genetics.[1] She introduced the game to the rest of the world in 1991, and it quickly became popular, especially among mathematicians.

Rules of the Game

The game of SET®is played with a deck of SET®cards. From the back of box:

> The object of the game is to identify "sets" of three cards. Each card is unique in its four features: **number** (1, 2, or 3); **symbol** (diamond, squiggle, or oval); **shading** (solid, striped, or open); and **color** (red, green, or purple).
> A "set" consists of three cards on which each feature is either *the same* on all of the cards, or *different* on all of the cards.

The Play. Twelve cards (the "stack") are dealt face up in a 3×4 array. The first person to see a set, calls "Set," and collects the three cards. The other players then check to see that indeed it is a bona fide set. Three cards are added to the stack and play continues. If ever the players agree that the stack contains no sets, then three cards are added to the stack for a total of 15. When a set is found in this new stack, those three cards are not replaced, unless there are no more sets. Play continues until the deck is exhausted, and the winner is the one with the most sets.

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[1]For a history of the game in Dr. Falco's own words, visit `http://www.setgame.com`.

Example 1. Can you find the six sets in this stack?
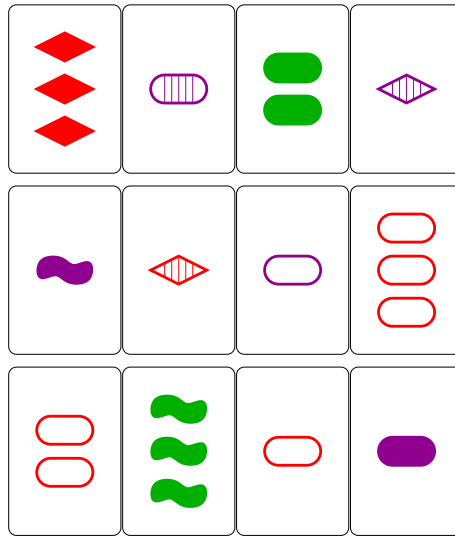
Solution. Here are the sets in the stack shown on the previous page:

(1)

(2)

(3)

(4)

(5)

(6)
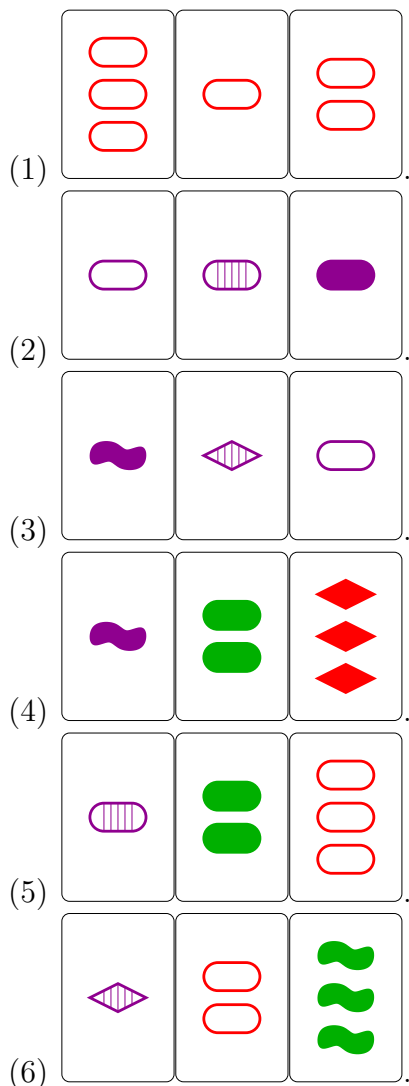
Question 1. A standard SET®deck, S, contains exactly one of each possible card. How many cards are in the deck?

Solution. There are 3 choices for each of the four features. Thus there are $3^4 = 81$ cards in the deck.

Note this suggests there is a correspondence between the deck of SET®cards, S, and the numbers $\{0, 1, 2, \ldots, 80\}$ when written in base-3. However, S has some more structure on it than simply being a collection of 81 elements. So we need to figure out the structure if we want to make this correspondence.

Question 2. How many sets are there in S?

Solution. If we choose any two cards, then the third card is uniquely determined. This is the Fundamental Theorem of Set, and will be proved shortly. Thus, there are 81 ways to choose the first card, and 80 ways to choose the second card, and this

determines the third card. However, once the cards are chosen, we can rearrange them in $3! = 6$ different ways. So the total number of sets is

$$\frac{81 \cdot 80}{3!} = 1080.$$

Theorem 1 (Fundamental Theorem of Set)**.** Given two cards, $x, y \in S$, there is a *unique* card $z \in S$ for which (x, y, z) is a set.

Question 3. Can you prove this?

Proof. Consider each of the four properties in turn. There are 3 choices for z in that property, *a priori.* However, if x and y are the same choice in that property, then z must be the same as well. If x and y are different choices in that property, then z must have the choice that is different from those of x and y. Therefore the cards x and y determine the third card z. □

Question 4. For each of the following pairs of cards, find the card that completes the set.
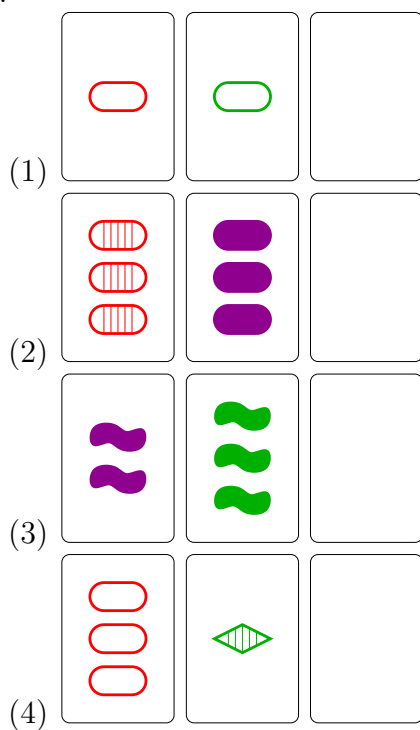
Multiplication

Thinking about the Fundamental Theorem of Set, we might consider that there is some sort of "multiplication" happening in S. In fact, we can define it this way: if x and y are two cards in S, define their *product* xy as the unique card for which (x, y, xy) is a set.

Example 2. In Question 4 we saw that
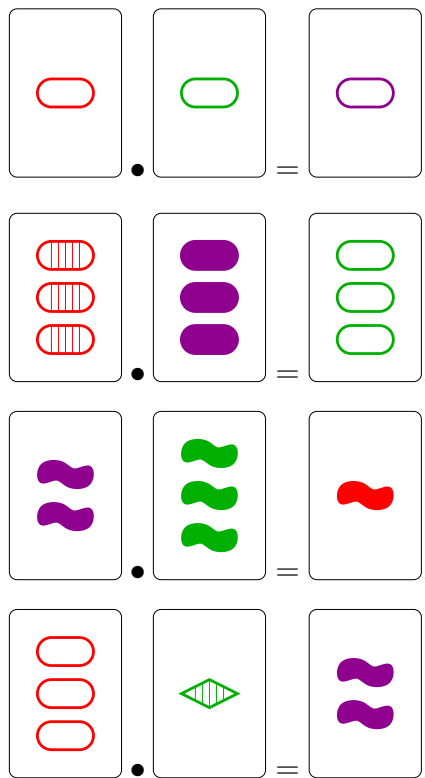
Question 5. What properties does this multiplication have? Is it commutative? associative? Is there an identity?

Solution. .

   (i) $xy = yx$,
  (ii) In general $x(yz) \neq (xy)z$,
 (iii) $x(xy) = y$,
 (iv) $xx = x$.

Question 6. We said before that there should be a correspondence $S \longleftrightarrow \{0, 1, 2, \ldots, 80\}$ by writing the numbers in base-3. What is the multiplication for these numbers?

Solution. Really, the set $\{0, 1, 2, \ldots, 81\}$ written in base-3 is $\mathbb{Z}_3^4$. In other words, we are talking about vectors of the form (x_1, x_2, x_3, x_4) where the $x_i \in \{0, 1, 2\}$. But how do we "multiply" two of these vectors. Well, first we will look at how to "multiply" two numbers $a, b \in \{0, 1, 2\}$

Given $a, b \in \{0, 1, 2\}$ define their *product* [2] $a \bullet b$ by

$$a \bullet b = 2(a + b) \mod 3.$$

You should check that if $a \neq b$ then $ab \neq a$ and $ab \neq b$; while if $b = a$ then $ab = a$.

Now back to S viewed as $\mathbb{Z}_3^4$. Define the multiplication

$$(x_1, x_2, x_3, x_4) \bullet (y_1, y_2, y_3, y_4) = (x_1 \bullet y_1, x_2 \bullet y_2, x_3 \bullet y_3, x_4 \bullet y_4).$$

Question 7. Compute the following products:

(1) $(1, 0, 0, 0) \bullet (1, 0, 0, 1) =$?
(2) $(0, 1, 0, 0) \bullet (0, 2, 0, 0) =$?
(3) $(2, 2, 2, 2) \bullet (0, 2, 2, 1) =$?
(4) $(0, 0, 0, 0) \bullet (1, 1, 1, 1) =$?

COUNTING

Question 8. Given a card $x \in S$, how many sets does x belong to?

Solution. There are 80 other cards in S. However, each set x belongs to uses two other cards. And two sets containing x can only intersect at x. Therefore there are 40 sets in S which have x as a member.

Question 9. Does this agree with what we said before?

Solution. It does: There are 40 sets for each card $x \in S$, and there are 81 cards total. However, each set belongs to three cards, so we have counted it thrice. Therefore, the total number of sets is

$$\frac{40 \cdot 81}{3} = 1080.$$

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[2]It might not look like it, but really this is just the *average* of a and b modulo 3. Why? Because multiplication by 2 modulo 3 is the same as dividing by 2! How cool is that? To see this, notice that $2 \cdot 2 = 1 \mod 3$. Therefore, $2 = 2^{-1} \mod 3$.

Question 10. What is the smallest stack such that every card must have a set in the stack? In other words, you want to deal off a number of cards, n, so that you *know*, without looking at the cards, that every one of those cards is in at least one set. How big must n be?

Solution. We can use the Fundamental Theorem of Set, and the fact that every card has 40 sets in S. Let x be any card. We can choose 1 card from each of the sets belonging to x without forming a set containing x. However, if we add just one more card, any card, then it must be the third card in a set with x. This tells us that it is possible to construct a stack with 41 cards so that the card x does *not* have a set.

Conversely, suppose x is in some stack of 42 cards given to us. Then there are 41 cards besides x. Therefore, there is at least one pair of cards y, z in that stack so that (x, y, z) is a set. So x has a set.

The answer is: given a stack of 42 cards, then every card *must* be in a set; given a stack of fewer than 42 cards, and this is not the case.

Question 11. Now, what is the *largest* stack with *no* sets? This one is a bit trickier.

Hint. First consider the game of Set with only 2 properties (9 cards in the deck), say **shape** and **number**. We can represent this game as a Tic-Tac-Toe board: How
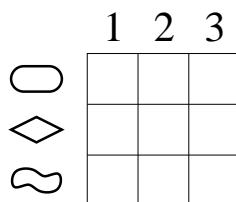

FIGURE 1. SET on 9 Cards with two Features.

many cards can we put in here? We want to avoid forming "lines" in the Tic-Tac-Toe sense. Note, however, that lines can wrap around the Tic-Tac-Toe board (really, it is Tic-Tac-Torus[3]). Now add a third property (a deck with 27 cards): And so on...
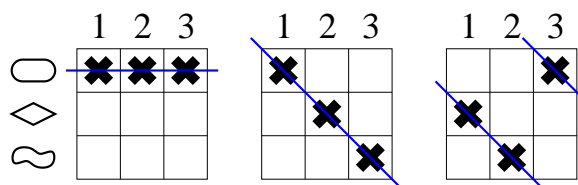
FIGURE 2. Some Sets on a Tic-Tac-Torus board.
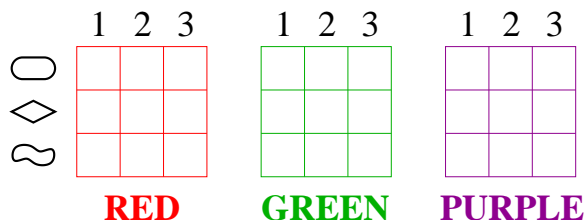
FIGURE 3. 3D Tic-Tac-Torus

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[3]In case you are wondering, Tic-Tac-Torus is even less fun than Tic-Tac-Toe—**X** always wins.
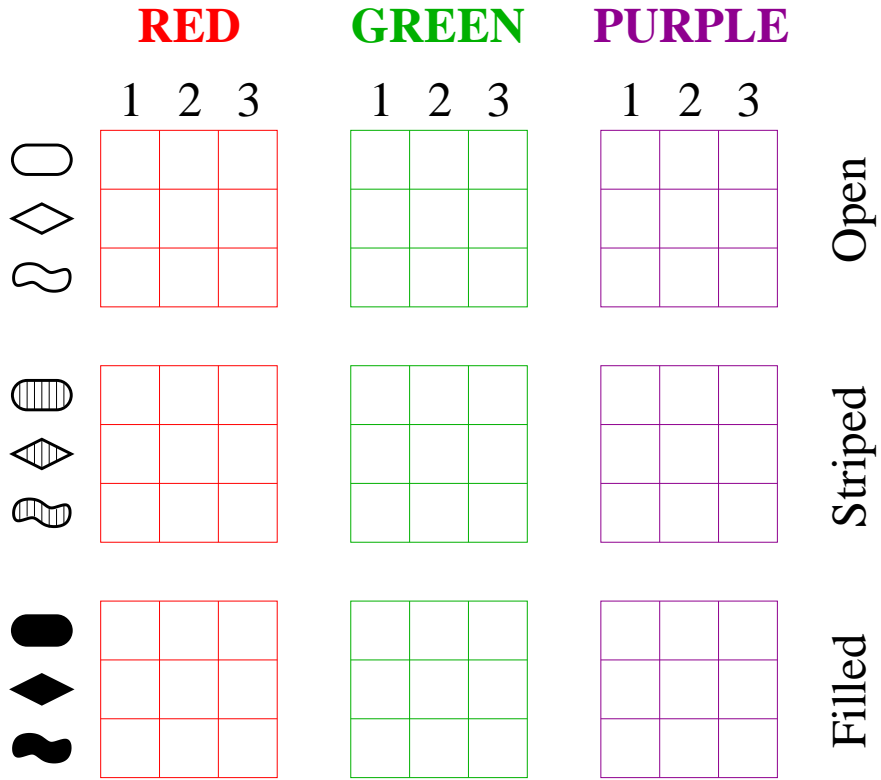
FIGURE 4. 4D Tic-Tac-Torus

Question 12. How many sets can we make with a stack of 4 cards? 5 cards? 6 cards? n cards? Use the 4D Tic-Tac-Torus to help organize your sets.

Question 13. What does a 5D Tic-Tac-Torus look like? What about a 6D Tic-Tac-Torus? What about ND?

Question 14. One day, Peter came to school and saw some students playing SET®, and they were down to the final stack, which looked like this:
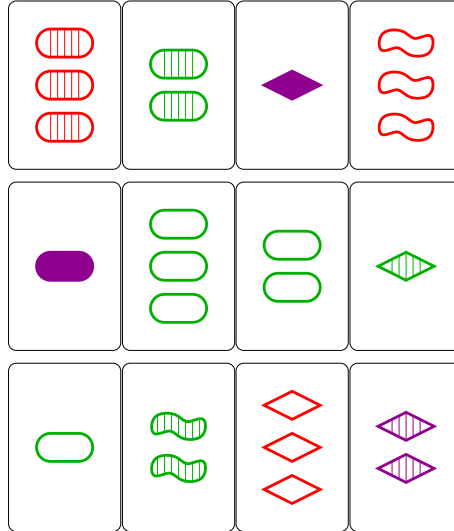
Peter knew that someone must have made a mistake earlier in the game. In other words, one of the players has 3 cards in their pile which they thought was a set, but in fact was not. How could Peter deduce this so quickly?

Question 15. Look at each of the following stacks and determine if they could be the final stack in the game.

(1) .

(2) .

(3) .
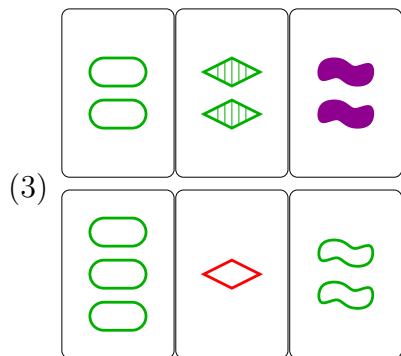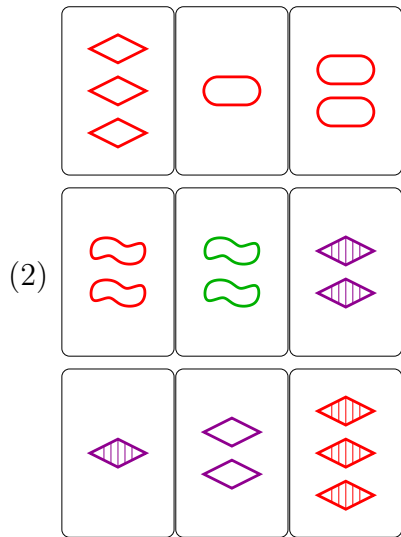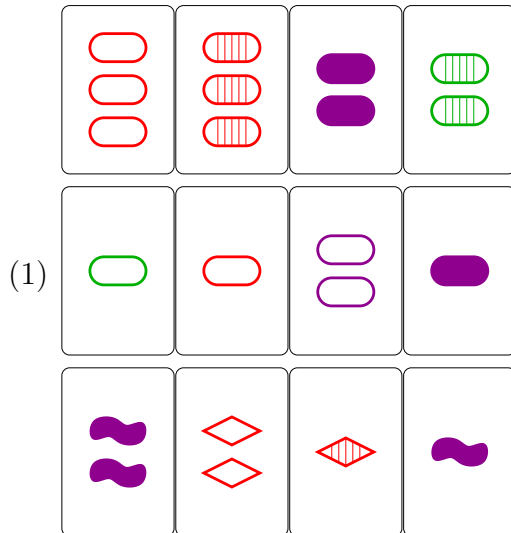
Question 16. One day you get out your game of SET®and start playing with your friends. All goes well until the very end, when you discover that one of the cards is missing! (your dog ate it!) So now your final stack has only 11 cards in it, instead of 12. Can you tell, just by looking at these 11 cards, which card your dog ate?
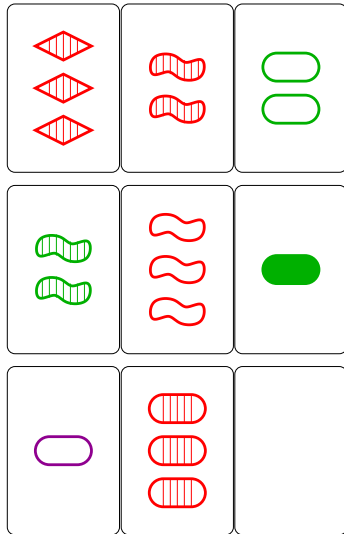
Question 17. For each of the following stacks, if possible, determine what the missing card is.

(3) This time, you didn't notice there was a missing card until after someone called "Set" on the stack of 11 and took three cards. You don't remember what those cards were. What is the missing card?

Open Question. It is easy to see that we can generalize the game of SET®by adding more Features, say k, or adding more options in each feature, say we used n shapes, colors, shadings, etc. Then the deck would have n^k cards. But can you answer the rest of the questions in this paper about our new n^k game? As far as I can tell, some of these questions, such as Question 11, have not been answered in general.