

DERANGEMENTS AND INCLUSION-EXCLUSION

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1. DERANGEMENTS

Problem 1. *At the beginning of class, we usually require that every student find the tag with his or her name on it. Now assume instead that every student chooses a name tag arbitrarily. What is the probability that at least one student chooses his/her own name? (Let us assume for example that there are 20 students in the class.)*

The same problem can be formulated in other similar ways, e.g.:

Problem 2 ([CD]). *Imagine that the Rice-Eccles stadium is completely sold out, but when ticket holders arrive, they choose seats at random. What is the probability that at least one person is seated in the seat indicated by his ticket? (The Rice-Eccles stadium has 45,000 seats.)*

Problem 3 (The hatcheck problem, [Ro]). *A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers of the hats. When customers return for their hats, the checker gives them back hats chosen randomly. What is the probability that no one receives the correct hat?*

A priori, would you expect a big difference in the probabilities for the two problems?

Following the approach of [CD]¹, let us do some experiments first and see what the probability may be. We will use decks of cards. From your deck of cards select all the spades and all the hearts. Place the spades in order:

A♠ 2♠ 3♠ 4♠ 5♠ 6♠ 7♠ 8♠ 9♠ 10♠ J♠ Q♠ K♠

Then shuffle the hearts and place them in line underneath, e.g.:

A♠ 2♠ 3♠ 4♠ 5♠ 6♠ 7♠ 8♠ 9♠ 10♠ J♠ Q♠ K♠
7♥ Q♥ 2♥ 4♥ 9♥ 3♥ A♥ 6♥ 5♥ J♥ K♥ 10♥ 8♥.

For every experiment, record if there is a match or not. In our example, there is one match, 4♠ with 4♥.

Then do the same experiments, but with only 6 cards.

Here is some experimental data from [CD], using a C-program and a random number generator. Below we say a deal is *good*, if there is at least one match. Let's say we do 100,000 deals in every case.

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¹This is a variant of a card game named *rencontres* played by really bored French people about 300 years ago. It is played with the entire deck, 52 cards. "Le problème de rencontres", posed by de Montmort at that time, asks what is the probability that there is no match? We will know the answer at the end of the class.

# cards in deck	Good deals
1	100,000
2	49,735
3	66,763
4	62,655
5	63,341
10	63,170
30	63,059

After these experiments, we can actually compute precisely the probability, when there are n cards and n is small. By listing all possible arrangements ($n!$), and counting which of them give matchings, we find the following table:

n	probability
1	$1/1 = 1$
2	$1/2 = .5$
3	$4/6 = .666\dots$
4	$15/24 = .625\dots$
5	$76/120 = .6333\dots$

Definition. A *derangement* is a permutation of objects, which leaves no object in its original position.

We will find a formula for the number of derangements of n objects.

2. THE PRINCIPLE OF INCLUSION-EXCLUSION

In order to solve this type of problems, we will use a very simple, yet powerful tool, the principle of inclusion-exclusion. The result is:

Theorem. Assume n finite sets A_1, A_2, \dots, A_n are given. Then

$$|A_1 \cup A_2 \cup \dots \cup A_n| = \sum_{1 \leq i_1 \leq n} |A_{i_1}| - \sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1} \cap A_{i_2}| + \sum_{1 \leq i_1 < i_2 < i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{n-1} \sum_{1 \leq i_1 < i_2 < \dots < i_{n-1} \leq n} |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_{n-1}}| + (-1)^n |A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_n}|$$

For example, if we have two sets A_1, A_2 , then

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|,$$

and if we have three sets A_1, A_2, A_3 , then

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

Here are some lists of problems that can be solved using this principle (selected from the bibliography).

1. How many positive numbers not exceeding 10000 are prime to both 7 and 11.
2. Assume there are 72 people for whom we have the following food preferences:

Chocolate	Broccoli	Tofu	#
		•	5
	•		8
	•	•	2
•			20
•		•	15
•	•		4
•	•	•	7

How many people don't like any of these foods?

3. There are 15 students in classroom. The teacher is not satisfied with the seating arrangement and demands that everyone move to a new seat. How many new configurations are possible?

4. Michael has a new cell phone and he's having difficulty remembering the new ten-digit phone number. His memory is strangely fragmented: he remembers that the second, fourth, and fifth digits are either 7 or 9, the third and tenth digits are either 2 or 4, there are two zeros in the number, and the sum of the digits is 42. Given this information, how many possibilities are there for Michael's phone number?

5. How many permutations of the 26 letters of the English alphabet do not contain any of the strings *fish*, *rat*, or *bird*.

6. How many integers can be expressed as a sum of two or more different members of the set $0, 1, 2, 4, 8, 16, 31$?

REFERENCES

- [CD] B. Conrey, T. Davis, *Derangements*, www.mathteacherscircle.org
 [Ro] K. Rosen, *Discrete Mathematics and its applications*, 6th edition, McGraw Hill.
 [T] P. Trapa, *The inclusion-exclusion principle*, www.math.utah.edu

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