ORDER

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We will consider some mathematical problems for which the common feature is the use of ordering between numbers.

1. Tallest of the shortest or shortest of the tallest?

Problem 1. *Nine students are positioned in three rows each containing three students. From each column pick the shortest student, then pick the tallest of these three (short) students, and tag him A. Next choose the tallest student in each row, then from these three (tall) students, pick the shortest and tag him B.*

Which one of A and B is taller (if different people)?

Proof. There are three cases:

1. (1) A and B are in the same column. Then, since in each column we picked the shortest student, A is shorter than B.
2. (2) A and B are in the same row. Then, since in each row we picked the tallest student, B is taller than A.
3. (3) A and B are neither in the same row, nor in the same column. Let C be a student who is in the same row as A and in the same column as B. Then by the same arguments as before, C is taller than A, but B is taller than C. This means that again B must be taller than A.

		A
B		C

In conclusion, the shorterst of the tallest is taller than the tallest of the shortest. □

Notice that in the argument above, we never used that there are 9 students, arranged 3×3. The same statement and proof hold if we have mn students arranged in m rows, with n students in every row. In fact, we may formalize this problem using numbers (each height is a number) as follows.

Problem 2. *Consider an $m \times n$ table of numbers such that at the (i, j)-position we have the number a_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$. (This is a matrix.) Then*

$$\min_{i=1}^{m} \max_{j=1}^{n} a_{ij} \geq \max_{j=1}^{n} \min_{i=1}^{m} a_{ij}. \tag{1}$$

Proof. This is the same as the solution for problem 1. □

A harder problem involving tables of integers is the following.

Date: September 24, 2008.

Problem 3. *In a* 10×10 *table are written the integers from* 1 *to* 100. *From each row we select the third largest number. Show that the sum of these numbers is not less than the sum of numbers in some row.*

Proof. Let's denote by $a_1, \ldots, a_{10}$ the ten numbers selected in this way, ordered as $a_1 > a_2 > \cdots > a_{10}$. We want show that the sum $a_1 + a_2 + \cdots + a_{10}$ is greater than the sum in some row. So we are looking for a row which potentially gives us a small sum. Since each of the numbers a_i is the third largest in its row, it makes sense to look at the row containing the smallest one, a_{10}. Let's see what the maximum possible sum is that this row can have. There are two numbers greater than a_{10} there, so at most, they are 100 and 99. Then there is a_{10}, and then there are 7 more numbers less than a_{10}. These 7 numbers can be at most $a_{10} - 1, a_{10} - 2, \ldots, a_{10} - 7$. So the maximum sum that the row containing a_{10} can have is

$$\text{maxsum(row)} = 100 + 99 + a_{10} + (a_{10} - 1) + \cdots + (a_{10} - 7) = 8a_{10} + 171. \quad (2)$$

On the other hand let's see how small the sum $a_1 + a_2 + a_3 + \cdots + a_{10}$ is. a_1 is the largest among these ten numbers: this means that in each row there are at most two numbers larger than a_1, so there are at most 20 numbers in the matrix larger than a_1. This implies $a_1 \geq 80$. Now for a_2: all numbers in the row containing a_1 could be larger than a_2, but then only 18 more numbers, so there are at most 28 numbers larger than a_2. So $a_2 \geq 72$. So far the sum is

$$\text{sum}(a_i) \geq 80 + 72 + (a_3 + a_4 + \cdots + a_{10}) = 152 + (a_3 + a_4 + \cdots + a_{10}). \quad (3)$$

Since in the equation (2) we have $8a_{10}$, it makes sense to compare the rest of a_i's with a_{10}. At the minimum, they are $a_9 \geq a_{10} + 1, a_8 \geq a_{10} + 2, \ldots, a_3 \geq a_{10} + 7$.

$$\text{sum}(a_i) \geq 80 + 72 + 8a_{10} + 1 + 2 + \cdots + 7 = 8a_{10} + 180. \quad (4)$$

From equations (4) and (2), we see that $\text{sum}(a_i)$ is greater than the sum of the row containing a_{10}.

□

A simpler case of the same problem, where the same kind of proof works is the following.

Problem 4. *In a* 4×4 *table are written the integers from* 1 *to* 16. *From each row select the second largest number. Show that the sum of these numbers is not less than the sum of numbers in some row.*

Is the same true if instead we choose the third largest number in each row?

Proof. For the second largest number chosen, the similar proof as before works. Let $a_1 > a_2 > a_3 > a_4$ be the numbers chosen. Then, following the same logic for the row containing a_4, we find that

$$\text{maxsum(row)} = 16 + a_4 + (a_4 - 1) + (a_4 - 2) = 3a_4 + 13. \quad (5)$$

Then we notice that $a_1 \geq 12$. So we have:

$$\text{sum}(a_i) \geq 12 + a_4 + a_4 + 1 + a_4 + 2 = 3a_4 + 15. \quad (6)$$

Then since $\text{sum}(a_i) > \text{maxsum(row)}$, we get the conclusion.

Now if we change the problem, and choose the third largest number in each row, the same proof would not work. If we do the calculations, we find that $\text{maxsum(row)} = 2a_4 + 30$, and $\text{sum}(a_i) \geq 2a_4 + 15$, so we cannot conclude anything

7	8	9	10
5	6	11	12
3	4	13	14
1	2	15	16

based on this. But we can construct a table for each the sum of the third largest from every row is less than the sum in each row:

The sum in each row is 34, and the sum of the "third largest" numbers is $8 + 6 + 4 + 2 = 20$. □

Question. Can you find a generalization of these examples for an $n \times n$ table?

2. Differences

Problem 5 (Putnam 1965)**.** *Consider the integers from* 1 *to* n*,* $n \geq 2$*. We want to order them in such a way that, except for the first integer at the left, every integer differs by* 1 *from some integer at the left of it. In how many ways can we do this?*

Proof. Let's look at some small values of n first to get an idea. We list all possible arrangements (there are $n!$ possibilities), and then select those which satisfy the condition.

If $n = 2$, then we have two arrangements (12) and (21) and they are both good.

If $n = 3$, then we have 6 possible arrangements and good ones are marked in boxes:

$$\boxed{(123)} \quad (132) \quad \boxed{(213)} \quad \boxed{(231)} \quad (312) \quad \boxed{(321)}. \tag{7}$$

If $n = 4$, there are 24 possible arrangements:

$$\begin{array}{cccccc} \boxed{(1234)} & (1243) & (1324) & (1342) & (1423) & (1432) \\ \boxed{(2134)} & (2143) & \boxed{(2314)} & \boxed{(2341)} & (2413) & (2431) \\ (3124) & (3142) & \boxed{(3214)} & \boxed{(3241)} & (3412) & \boxed{(3421)} \\ (4123) & (4132) & (4213) & (4231) & (4312) & \boxed{(4321)} \end{array} \tag{8}$$

We notice that for $n = 2$ there are 2 good arrangements, for $n = 3$, there are 4, and for $n = 4$, there are 8. So we may conjecture that for an arbitrary n, there are 2^{n-1} good arrangements.

But if we look more closely, we see another pattern, all good arrangements end in 1 or n. Let's try to prove this first by induction. Assume that for $n-1$, all good arangements end in 1 or $n-1$. Now let's consider a good arrangement for n, and assume it doesn't end in 1. So we want to prove it ends in n. Notice first that $n-1$ cannot be at the right of n, because then the largest possible number at the left of n if $n-2$, so n does not have the property. Therefore $n-1$ must be at the left of n. Then by removing n we must get a good arrangement for $n-1$, which, by induction, must end in $n-1$, because the original arrangement didn't end in 1. So n must have been the last entry.

OK, so now by induction, we know that every good arrangement must end in n or in 1, moreover we may assume there are 2^{n-2} good arrangements for $n-1$. Notice further that from any good arrangement which ends in n we can construct a good one which ends in 1, by replacing every j in the arrangement with $n+1-j$.

For example, (3214) gives (2341) by replacing j with $5-j$. So it is sufficient to show that there are 2^{n-2} good arrangements that end in n. This follows by induction by adding an n at the end of every good arrangement for $n-1$. □

Problem 6. *Assume $n \geq 3$. Consider $2n$ distinct positive integers $a_1, a_2, \ldots, a_{2n}$ not exceeding n^2. Prove that there are at least three differences $a_i - a_j$ that are equal.*

Proof. Again, let's begin with an example, $n = 3$. From the numbers $1, 2, 3, \ldots, 9$ we should pick 6 different arbitrary ones, and see that there are at least 3 equal differences.

For example, let's say we pick first 1, then 2 and 3;

$$\boxed{1}, \boxed{2}, \boxed{3}, 4, 5, 6, 7, 8, 9. \tag{9}$$

The differences $3-2 = 2-1 = 1$. If next we pick 4, since $4-3 = 1$, we would have our three equal differences. So let's pick 5.

$$\boxed{1}, \boxed{2}, \boxed{3}, 4, \boxed{5}, 6, 7, 8, 9. \tag{10}$$

Now, we can't pick 6 next because $6-5 = 1$. So we pick 7.

$$\boxed{1}, \boxed{2}, \boxed{3}, 4, \boxed{5}, 6, \boxed{7}, 8, 9. \tag{11}$$

Now we have two differences equal to 1, and two differenced equal to 2, and we have one more number to pick among 8 or 9. If we pick 8, then $8-7 = 1$, if we pick 9, then $9-7 = 2$, so we fought in vain...

This examples gives the idea for the proof: we look at the numbers ordered, and we consider differences of consecutive numbers. So let's assume the order is

$$a_1 < a_2 < a_3 < \cdots < a_{2n}, \tag{12}$$

(all are numbers between 1 and n^2), and write the differences between consecutive numbers:

$$a_2 - a_1, a_3 - a_2, \ldots, a_{2n} - a_{2n-1}. \tag{13}$$

These differences are all positive numbers. Their total sum, call it Δ, is

$$\Delta = (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{2n} - a_{2n-1}) = a_{2n} - a_1 \leq n^2 - 1. \tag{14}$$

On the other hand, if we assume that no three differences are the same, then among the differences there could be at most two 1's, at most two 2's,..., at most two n's. This means that

$$\Delta \geq 1 + 1 + 2 + 2 + \cdots + n + n = 2(1 + 2 + \cdots + n) = n(n+1) = n^2 + n. \tag{15}$$

Since $n^2 + n > n^2 - 1$, this is a contradiction. □

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