KNOTTY MATH CIRCLE CONTEST

March 12, 2003

(1) *YOU'RE THE BOSS*

You are the boss of a little company and are trying to organize the work-shifts schedules of your employees. As you do that, you need to keep in mind the constraints that the bizzarre personalities of your workers give you:

- Alice and Ben can't stand Charlie and David;
- Charlie and David don't tolerate Ernst and George;
- George hates Frank and Ben;
- Ernst doesn't like Alice, Ben nor George;
- Frank only likes Ernst and Hubert;
- Hubert is really grumpy and can only work with George.

1a) What is the smallest number of workshifts you need?

1b) How many different schedules can you organize using 4 shifts?

Now you hire a new guy, Ivan, who claims to be really easy going. He is so mellow that even Frank and Hubert don't have a problem working with him (although they still can't work with each other).

1c) How many 4 shifts schedules can you now organize?

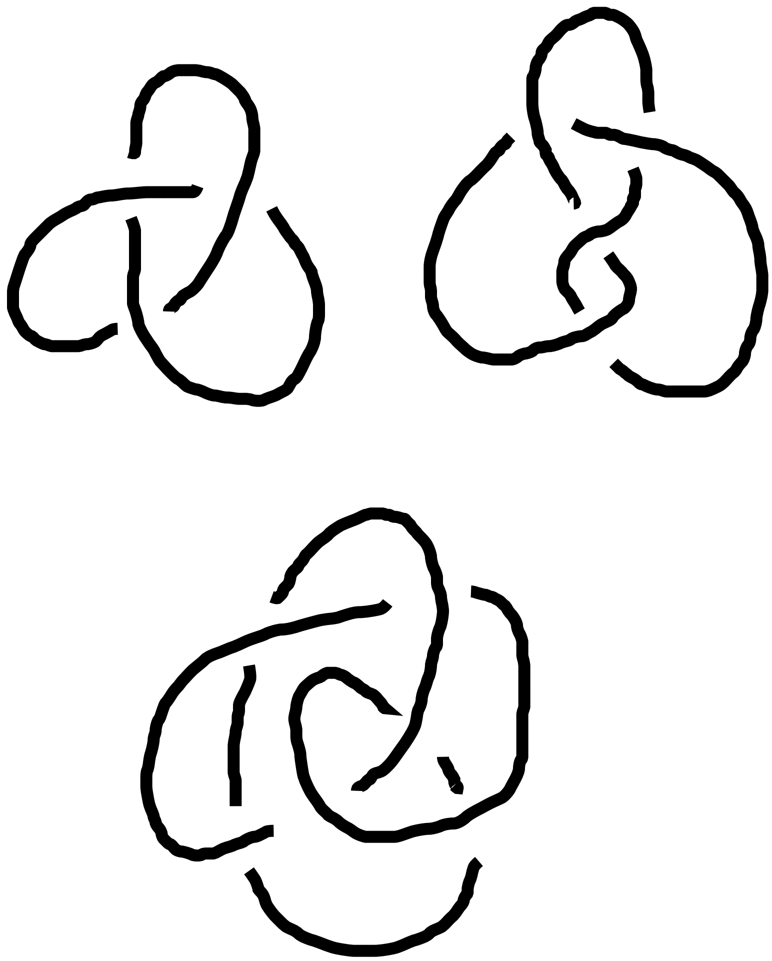
However, Ivan falls in love with Alice, so for the sake of productivity it's better to not schedule them together.

1d) How many 4 shifts schedules can you organize now?

WARNING: HARD QUESTION COMING UP!!

1e) Suppose now that you don't want to scehdule Ivan and Ben toghether either. Notice that the number of 4 shifts schedules doesn't change. Can you explain why using graph theory?

Problem 2) Show that exactly two of the three diagrams below represent the same knot.



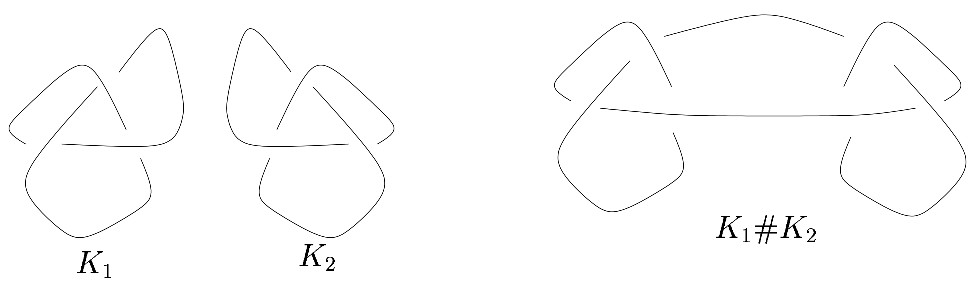
(3) *CUTTING, PASTING AND COLORING*

Recall that a knot K is tricolorable if one can color the strands (i.e. pieces) of a planar projection of K (i.e. a picture of K squished onto the plane) with three colors such that

- one uses all three colors and
- at each crossing all the strands are either distinct colors or all the same color.

Several weeks ago, we saw that tricolorability remains unchanged if you take different planar projections of a knot (i.e. wiggle the knot and then squish it into a plane).

There is an operation on knots which we did not discuss. Roughly speaking, this operation is splicing two knots K_1 and K_2 together to create a new knot, called $K_1\#K_2$. To do this operation, you first cut each knot K_1 and K_2 someplace and then you join the open strands to create one knot. When you are joining the two open strands of the knots, you must be careful not to introduce any new crossings. To see an example, look at the picture of the splicing of two trefoil knots below.



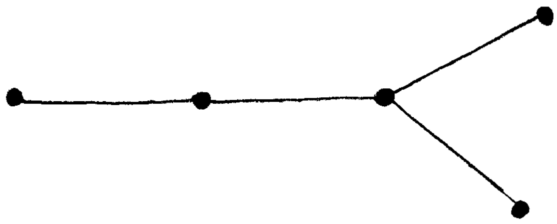
This question will ask about the tricolorability of the splicing of two knots.

3a) Suppose you have two knots K_1 and K_2, both of which are tricolorable. Is the splice $K_1\#K_2$ also tricolorable?

3b) Suppose you have two knots K_1 and K_2, where this time K_1 is tricolorable and K_2 is *not*. Is the splice $K_1\#K_2$ tricolorable?

3c) Suppose you have two knots K_1 and K_2, neither of which is tricolorable. Is it possible that the splice $K_1\#K_2$ is tricolorable? (Warning: this part is more difficult than the other two parts of this question.)

Problem 4) a) Compute the chromatic polynomial of the following graph:



b) Recall that any graph that does not contain a cycle is called a *tree*. Find the chromatic polynomial of a tree with n vertices.

c) Find a graph G whose chromatic polynomial is

$$P_G(x) = x(x-1)^5 - x(x-1)^4.$$

Hint: Recall the formula $P_G(x) = P_{G-e}(x) - P_{G\setminus e}(x)$, where $G-e$ is the graph formed by removing the edge e from G, and $G \setminus e$ is the graph produced by contracting the edge e.