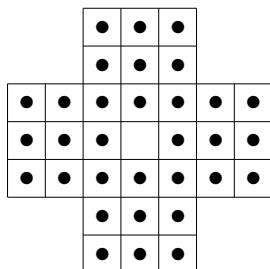# Peg Solitaire and Group Theory

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February 2006

## 1 A description of Cross Peg Solitaire

The board is cross shaped and contains of 35 holes. A peg might be placed in each hole. The starting configuration is a full board except for an empty place in the center.



A peg is allowed to jump vertically or horizontally over an adjacent peg to an empty hole. The peg that was jumped over is removed, decreasing the number of pegs on the board by one.
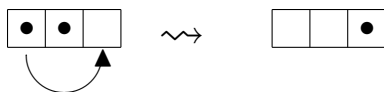


The object of the game is to invert the starting configuration, i.e. to end up with one peg in the center of the board. According to the game brochure (Milton Bradley Co., 1986), whoever succeeds in leaving the last peg in the center is a *genius*. Anyone who leaves a single peg elsewhere is an *outstanding player*.
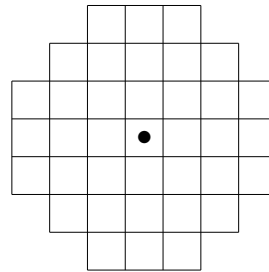
## 2 A variation of the game: Octagon Solitaire

We can alter the game by changing the starting position, ending position or the structure of the board. The following version of the game was manufactured and sold in Israel. The

board is now in the shape of an octagon.
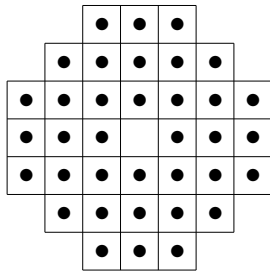


The starting configuration          The winning configuration

The game's brochure didn't include a solution. Alternatively, the manufacturer offered a prize to anyone able to come up with one. No matter how generous the prize actually was, the manufacturer wasn't taking any chances by offering it. The game is insolvable!

Let's take a moment to reflect here. How can we know for sure that no matter which way we proceed we'll never be able to end up in the winning position?

In this case we can make the following rough estimate: Any peg can move in 4 directions or less. At any given time there are no more than 38 pegs on the board. Hence, at any point of the game we have no more than $4 \cdot 38 = 152$ options for legal moves. Since we lose a peg with every move and we start with 38 pegs, a single game will consist of 37 moves or less. Therefore, the number of possible games is less than $(152)^{37}$.

Therefore, in order to show that no solution exists all we have to do is check all possible games and verify that none of them is a winner. The problem is that $(152)^{37}$ is an astronomical number and it is quite impractical to go over so many games, even with the aid of a computer. Moreover, oftentimes in Math or Physics we want to affirm the impossibility of a process or the existence of an object, and there usually the number of possibilities there is not finite, so such an approach will never work.

# 3    Aside: No solution to the general quintic polynomial

You might have seen that the roots of the quadratic polynomial: $f(x) = ax^2 + bx + c$ are given by $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. This means that $f(x) = 0$ only when we plug in $x_1$ or $x_2$ (when they exist). The roots $x_1$ and $x_2$ are given by a formula involving the coefficients of $f(x)$ and the operations: addition, subtraction, multiplication, division and taking square roots. The solution to the quadratic equation was probably known to the ancient babylonians as

early as 1600 BC. In 1545, Girolamo Cardano, an Italian physician, published his *Ars Magna* which contained a solution to the cubic polynomial (disclosed to him by a mathematician called Fontana). This volume also contained a method, due to Ludovico Ferrari, of solving the quartic polynomial (degree 4) by reducing it to a cubic. All the formulae discovered had the striking property that the expression for the roots was built up from the coefficients by repeated addition, subtraction, multiplication, division and extraction of roots. These became known as *radical* expressions.

Since all equations of degree $\leq 4$ were now solved it was natural to assume that the quintic (degree 5) equation could also be solved by a radical expression. After a long period of failed attempts and slow progress on this question, Niels Abel proved in 1824 that the general quintic equation is insolvable by radicals. In other words: **There are some polynomials of degree 5 who's roots are not radical expressions of the polynomial's coefficients!** (Clearly a counting argument could not help us here, since there is an infinite number of such expressions).

The final analysis of which polynomials are solvable by radicals and which aren't was done by a young Frenchman, Évariste Galois in 1832, though it took some 20 years for his work to be recognized. The beautiful theory he developed to solve this problem is still inspirational today in many branches of mathematics.

# 4 The insolvability of Octagon game

Lets "color" our board with the numbers 0 and 1 as in the figure.

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
|   |   | 1 | 1 | 0 |   |   |
|   | 1 | 1 | 0 | 1 | 1 |   |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 |
|   | 1 | 0 | 1 | 1 | 0 |   |
|   |   | 1 | 1 | 0 |   |   |

For a configuration $C$ let $N_C$ be the sum of "colors" of the filled places. For example, $N_{\text{start}}^{\text{oct}} = 24$ (we do not count the empty 1 in the center).

```
          1 1 0
      1 1 0 1 1
    1 1 0 1 1 0 1
    1 0 1 1 0 1 1
    0 1 1 0 1 1 0
      1 0 1 1 0
        1 1 0
```

Notice that preforming an elementary move might change $N$, or leave it unchanged as demonstrated by the following examples:

$$\boxed{1\ 1\ 0} \quad \rightsquigarrow \quad \boxed{1\ 1\ 0} \qquad\qquad \boxed{1\ 0\ 1} \quad \rightsquigarrow \quad \boxed{1\ 0\ 1}$$

$$N \text{ decreases by } 2 \qquad\qquad\qquad\qquad N \text{ doesn't change}$$

However, the parity of $N_C$ remains the same throughout, since moves either leave $N$ unchanged or decrease it by 2. Let's take a moment to digest this: the parity of $N_C$, $p(N_C)$ is an <u>invariant</u> of the game, which means it remains constant (while $C$ changes) because as we've seen no legal move can change it.

$N_{\text{start}}^{\text{oct}} = 24$ which is even, and since the center is marked by 1, $N_{\text{win}}^{\text{oct}} = 1$ which is odd. So we cannot get from one configuration to the other. We cannot win this game!

# 5   Introduction to Groups

We will now show the insolvability of the octagon game by a different method — using groups.

A group is a collection of reversible actions that we can carry out one after the other. Consider the actions you take to get dressed in the morning:

- put a sock on the right leg.

- remove sock from the left leg.

- put on both shoes.

- put on a pair of pants.
  ⋮

Sometimes, the order of the actions matters and sometimes it doesn't. Putting on your right sock first and then your left one, or doing it the other way around, doesn't really change the end result. However, putting on your underwear first and then your pants is different than putting on your pants first and then your underwear! It seems superman hasn't realized this yet, but he is an alien after all. Here's another example, "Don't drink and drive" is not quite the same as "Don't drive and drink".

Two elements (actions) in a group are said to <u>commute</u> if the order in which they are carried out doesn't matter. A group in which every two actions commute, is called <u>commutative</u> or <u>abelian</u> (named after Abel, whom we've already met in section 3).

# 6   Klein's Group

Suppose we have two light bulbs, and two buttons. Pushing button $A$ lights the left bulb, and another push turns it off. We denote the action of pressing $A$ by $a$. Pushing button $B$ lights the right bulb, another turns it off, this will be denoted $b$. $c$ is pressing the two buttons at once, and $e$ is not doing anything at all.

Our group, $K$, will consist of 4 elements: $a, b, c$ and $e$. Doing $a$ and then $a$, denoted $a \cdot a$ or just $a^2$, is like doing nothing at all. We express this by the equation $a^2 = e$. The same goes for $b$: $b^2 = e$. Doing $a$ and then $b$ amounts to lighting both bulbs so $a \cdot b = c$. We completed the multiplication table for the group $K$ below (multiplying two elements just means doing one after the other from left to right).

$$
\begin{array}{c|cccc}
 & e & a & b & c \\ \hline
e & e & a & b & c \\
a & a & e & c & b \\
b & b & c & e & a \\
c & c & b & a & e
\end{array}
$$

$\leftarrow$ do first

do second $\nearrow c$

Clearly, $K$ is an abelian group.

# 7 Back to the game

We shall now color our board with elements of $K$, instead of 0s and 1s.

| | | a | b | c | | |
|---|---|---|---|---|---|---|
| | a | b | c | a | b | |
| a | b | c | a | b | c | a |
| b | c | a | b | c | a | b |
| c | a | b | c | a | b | c |
| | b | c | a | b | c | |
| | | a | b | c | | |

For a configuration of the board $C$ let $M_C$ be the product of all markings of full squares, where the multiplication follows the rules of the table above. Note that we can multiply them in any order we choose since $K$ is abelian. So, $M_{\text{start}}^{\text{oct}} = a^{12} \cdot b^{12} \cdot c^{12} = e$. Another observation is that $M_C$ is invariant, notice that elementary moves don't change the product!

| a | b | c |
|---|---|---|

$\rightsquigarrow$

| a | b | c |
|---|---|---|

| b | c | a |
|---|---|---|

$\rightsquigarrow$

| b | c | a |
|---|---|---|

$M_{\text{start}}^{\text{oct}} = e$ but none of the squares is marked $e$. So not only can we never win this game and get one peg in the center, but we can never reach any one peg configuration!

# 8 Analysis of the cross game using Klein's group

Remember that according to the game's manufacturer, whoever ends up with one peg not in the center is an outstanding player? In this section we will see that anyone who ends up with one peg, not in the center cannot qualify for that title. He is merely very very lucky. Color the board with elements of Klein's group as below:
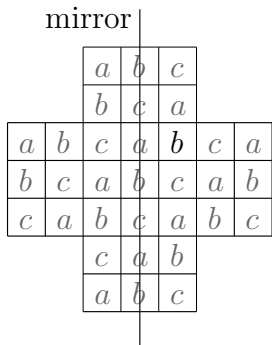
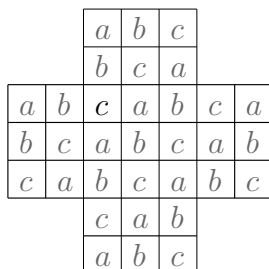| | | a | b | c | | |
|---|---|---|---|---|---|---|
| | | b | c | a | | |
| a | b | c | a | b | c | a |
| b | c | a | b | c | a | b |
| c | a | b | c | a | b | c |
| | | c | a | b | | |
| | | a | b | c | | |

$M_{\text{start}}^{+} = b$. Therefore, if we end up with one peg on the board it will be marked $b$. But there is another obstruction. If we can reach the configuration (1) below, we can also reach its

mirror image, numbered (2) - just mirror image the moves!

(1)

mirror

|   |   | a | b | c |   |   |
|---|---|---|---|---|---|---|
|   |   | b | c | a |   |   |
| a | b | c | a | b | c | a |
| b | c | a | b | c | a | b |
| c | a | b | c | a | b | c |
|   |   | c | a | b |   |   |
|   |   | a | b | c |   |   |

(2)

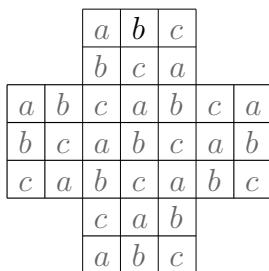|   |   | a | b | c |   |   |
|---|---|---|---|---|---|---|
|   |   | b | c | a |   |   |
| a | b | c | a | b | c | a |
| b | c | a | b | c | a | b |
| c | a | b | c | a | b | c |
|   |   | c | a | b |   |   |
|   |   | a | b | c |   |   |

But by our previous consideration, the configuration on the right is impossible, since it is marked by $c$. Hence the configuration on the left is also unattainable. We can thus rule out all one peg configurations except for the winning one and:
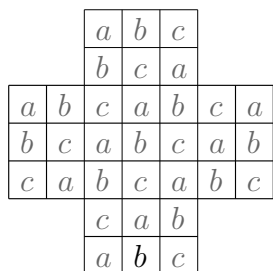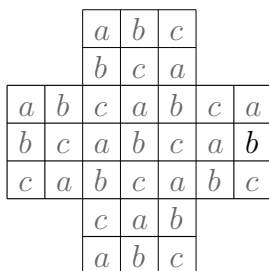
(i)

|   |   | a | b | c |   |   |
|---|---|---|---|---|---|---|
|   |   | b | c | a |   |   |
| a | b | c | a | b | c | a |
| **b** | c | a | b | c | a | b |
| c | a | b | c | a | b | c |
|   |   | c | a | b |   |   |
|   |   | a | b | c |   |   |

(ii)

|   |   | a | **b** | c |   |   |
|---|---|---|---|---|---|---|
|   |   | b | c | a |   |   |
| a | b | c | a | b | c | a |
| b | c | a | b | c | a | b |
| c | a | b | c | a | b | c |
|   |   | c | a | b |   |   |
|   |   | a | b | c |   |   |

(iii)

|   |   | a | b | c |   |   |
|---|---|---|---|---|---|---|
|   |   | b | c | a |   |   |
| a | b | c | a | b | c | a |
| b | c | a | b | c | a | b |
| c | a | b | c | a | b | c |
|   |   | c | a | b |   |   |
|   |   | a | **b** | c |   |   |

(iv)

|   |   | a | b | c |   |   |
|---|---|---|---|---|---|---|
|   |   | b | c | a |   |   |
| a | b | c | a | b | c | a |
| b | c | a | b | c | a | **b** |
| c | a | b | c | a | b | c |
|   |   | c | a | b |   |   |
|   |   | a | b | c |   |   |
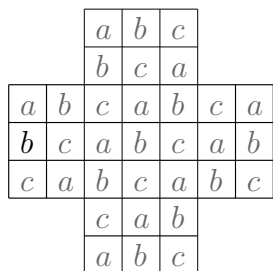
Now comes the punch line: suppose we've arrived at configuration (i) then the previous one must have been:

| | | | a | b | c | |
|---|---|---|---|---|---|---|
| | | | b | c | a | |
| a | b | c | a | b | c | a |
| b | **c** | **a** | b | c | a | b |
| c | a | b | c | a | b | c |
| | | | c | a | b | |
| | | | a | b | c | |

But we could have easily gone from that configuration to a winning one. Therefore, whoever reaches configuration (i) must not have comprehended the rules of the game, hence must have been very lucky to end up where he did.

# 9 Warm-up

Get from the following starting positions, to the winning configuration.
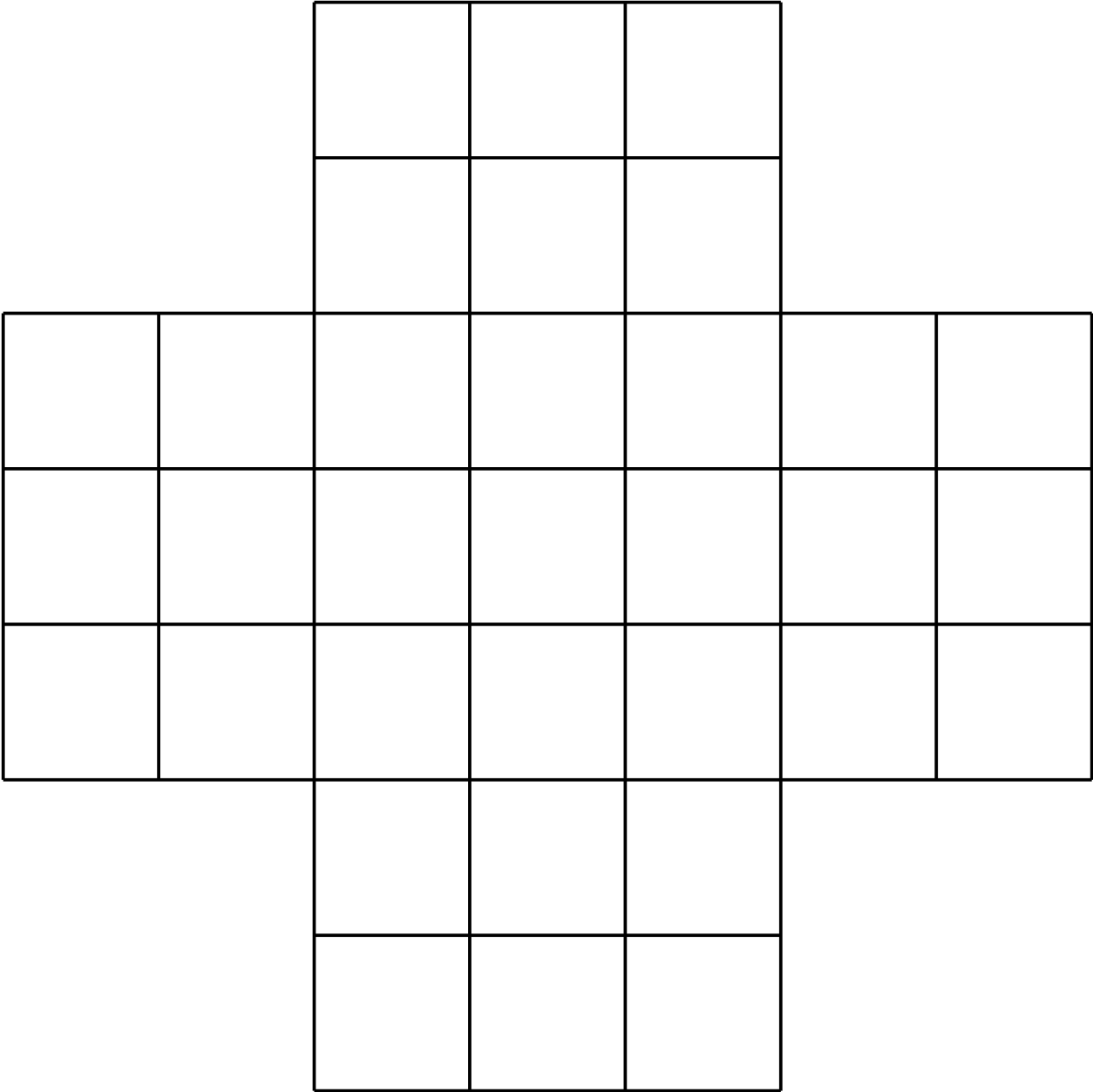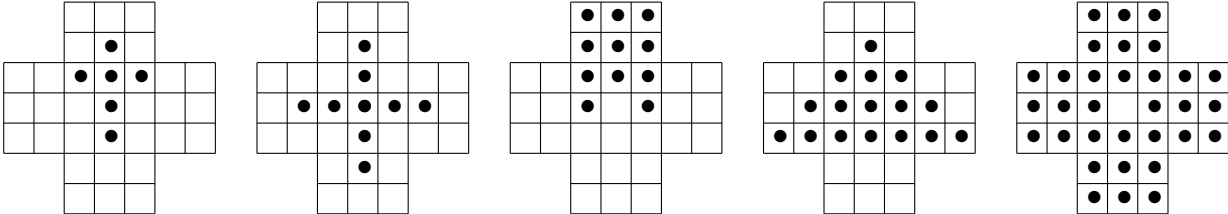
# 10 Exercises: The First Batch

1)  a. What is $N_{\text{start}}$ for the cross shaped game with the following coloring? What is the parity of $N_{\text{start}}$?

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
|   |   | 1 | 1 | 0 |   |   |
|   |   | 1 | 0 | 1 |   |   |
| 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 1 | 1 | 0 |
|   |   | 0 | 1 | 1 |   |   |
|   |   | 1 | 1 | 0 |   |   |

  b. Notice that legal moves still leave $p(N_C)$ unchanged. Suppose you end up with only one peg. In which squares can it be?

2) Would the following coloring of the board still leave $p(N_C)$ invariant? Check how the contribution of a triplet changes after doing an elementary move.

|   |   |   |   |   |   |   |
|---|---|---|---|---|---|---|
|   |   | 0 | 1 | 0 |   |   |
|   |   | 1 | 0 | 0 |   |   |
| 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 0 | 0 | 1 | 0 |
|   |   | 0 | 0 | 1 |   |   |
|   |   | 0 | 1 | 0 |   |   |

3) Prove that the following games are insolvable.
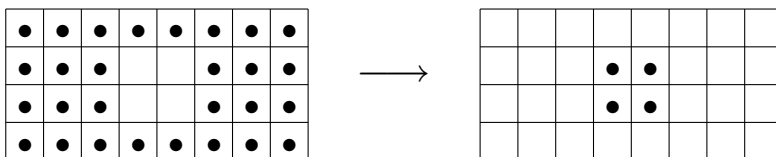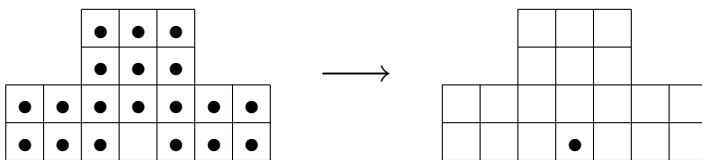


4) Consider the board, obtained from an $8 \times 8$ chess-board by removing two corner squares

on the same diagonal. Is it possible to completely cover this board with domino pieces? Each domino piece covers two squares.

5) Can you improve the bound on the number of possible games made in section 2?

# 11 Exercises: A Second helping

1. Consider the following group of actions on the real line: for any integer number $n$, $f_n(x) = x + n$. You can think of $\mathbb{R}$ as a long (infinite) piece of string with a stationary red bead at 0. $f_n$ shifts the string $n$ inches to the right if $n$ is positive and $|n|$ inches to the left if $n$ is negative.

   a) Describe $f_0$.

   b) Describe $f_3 \cdot f_2$ (don't get confused this is not regular multiplication! Here $f_3 \cdot f_2$ means do $f_3$ and then do $f_2$).

   c) Describe $f_2 \cdot f_3$ (remember only the end result matters).

   d) Describe $f_1 \cdot f_{-1}$.

   e) What is $f_n \cdot f_m$ for any integer $n$ and $m$?

   This group should be familiar to you by now. If we forget that the $f_n$s are actions and just keep in mind the way they interact in the group we get the group of integer numbers, and the multiplication between actions is simply addition of integers (Weird, isn't it?).

2. Suppose we have three drawers marked 1,2 and 3, and three balls marked $a$, $b$ and $c$. We start with $a$ in 1, $b$ in 2 and $c$ in 3. Let $f$ be the action of switching the balls in drawers 1 and 2. Let $g$ be the action of taking the ball in drawer 1 to drawer 2, the ball in drawer 2 to drawer 3 and the ball in drawer 3 to drawer 1.

   a) Describe $f \cdot g$.

   b) Describe $g \cdot f$.

   c) Describe $f^2$ and $f^3$.

   d) Describe $g^2$ and $g^3$.

   This is a non-abelian group, which contains 6 elements (actions). It is called: the symmetric group on 3 letters, denoted $S_3$.