TROPICAL ALGEBRA

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1 Introduction

Recall that $x^a \cdot x^b = x^{a+b}$. If we were to worry only about exponents then it looks like multiplication behaves more like addition. What does addition $(x^a + x^b)$ act like? Let's consider the example of polynomials.

Let $f(x) = x + 2$ and $g(x) = 3x^2 + x$. Let's call the highest degree of in a polynomial to be the **degree** of that polynomial, and the lowest degree to be the **order**. Then what are the degrees and orders of f and g?

1. $degree(f) =$
2. $degree(g) =$
3. $order(f) =$
4. $order(g) =$

What about $f(x) \cdot g(x)$? We have

$$f(x) \cdot g(x) = (x + 2)(3x^2 + x) = 3x^3 + x^2 + 6x^2 + 2x = 3x^3 + 7x^2 + 2x,$$

and so

1. $degree(f \cdot g) =$
2. $order(f \cdot g) =$

Can you see a general relationship between the orders of f and g and $f \cdot g$?

What about $f(x) + g(x)$? Since

$$f(x) + g(x) = x + 2 + 3x^2 + x = 3x^2 + 2x + 2,$$

we conclude

1. $degree(f + g) =$
2. $order(f + g) =$

Can you again see a general relationship between the orders of f and g and $f+g$?

What is the point of all of this? If we are only worried about what happens to the exponents of expressions like f and g when they are multiplied or added together, then it turns out that multiplication really looks like addition, and addition really looks like taking a minimum or a maximum. We are just going to stick with minimums for the rest of the time (but you could just as easily do all of this for maximums).

Let's actually formally define our new tricks for multiplication and addition, they will be called **tropical multiplication** and **tropical addition**.

$$a \odot b = a + b$$
$$a \oplus b = \min\{a, b\}$$

The above set of rules defines **min-plus algebra**, also called **tropical algebra**. To warm up, let's make sure we know how this actually works.

1. $3 \odot 1 =$
2. $3 \oplus 1 =$
3. $-1 \oplus [5 \odot (3 \oplus 1)] =$
4. $-1 \oplus [5000 \odot (300 \oplus 100000)] =$

Given these new operations, it's natural to ask to what extent they obey the usual rules of arithmetic. This is taken up in the next few sections.

2 Basic properties

The usual operations of arithmetic ($+$ and $\cdot$) are commutative, and it's easy to see their tropical counterparts ($\oplus$ and $\odot$) are as well. Associativity of $\odot$ is obvious

$$a \odot (b \odot c) = (a \odot b) \odot c,$$

since it holds for $+$. It requires just a moment's more thought to check that the operation $\oplus$ is also associative,

$$a \oplus (b \oplus c) = (a \oplus b) \oplus c.$$

A slightly trickier one is the distributive law. Is it always true that

$$a \odot (b \oplus c) = (a \odot b) \oplus (a \odot c)$$

for all a, b and c? Try a few examples and decide for yourself.

3 Identities

In regular arithmetic, if a is some number then what is $a \cdot 1$? It's a, of course. This is because 1 is the multiplicative identity, which is to say that multiplication by 1 returns what you started with. In tropical arithmetic, is there a number like this?

$$\text{If } 3 \odot a = 3, \text{ then } 3 + a = 3. \text{ So } a = 0.$$

Does this work for every number (not just 3)? If so then we would need $x \odot 0 = x$ for every x. Is it true? Yes, of course it is. The conclusion is that with respect to $\odot$, 0 behaves like 1 behaves with respect to ordinary multiplication:

$$a \odot 0 = a \text{ for all } a.$$

It's natural to ask the corresponding question for the $\oplus$ operation. Is there a tropical additive identity? A moment's thought shows that with respect to $\oplus$, ∞ behaves like 0 behaves with respect to $+$:

$$\infty \oplus a = a \text{ for all } a.$$

4 Inverses

In regular math every number has an additive inverse: For every real number a there is another number $-a$ so that $a + (-a) = 0$ Also, every number has an multiplicative inverse: For every nonzero number a there is a number $a^{-1} = \frac{1}{a}$ so that $a \cdot a^{-1} = 1$.

We can ask the same questions for $\oplus$ and $\odot$. Let's start with $\odot$. Recall that in the previous section 0 is the identity element for $\odot$. They question is thus: given any a, can we find another number (say b) so that $a \odot b = 0$? This is easy since $a \odot (-a) = 0$. So the $\odot$-inverse of a always exists and it is always $-a$. In fact, for this reason, it is sometime convenient to write $a^{-1} = \frac{1}{a} = -a$ in tropical arithmetic.

Can we find inverses for $\oplus$? Since we decided that ∞ is the identity for $\oplus$, the question becomes: for all a does there exist another number (say b) so that $a \oplus b = \infty$? You decide.

EXERCISES 1

Using the rules of tropical arithmetic discussed above, find all possible values of x which satisfy the given equations.

1. $x \odot 2 = 3$
2. $x^2 \odot 2 = 14$
3. $x^2 \odot x = 9$
4. $x \oplus 2 = 1$
5. $x \oplus 2 = 2$
6. $x \oplus 2 = 3$
7. For $x \oplus 2 = a$, can you find a general way to solve for x in terms of a?
8. $x^2 \oplus 3 = 1$
9. $x^2 \oplus 3 = 3$
10. $x^2 \oplus 3 = 5$
11. For $x^2 \oplus 3 = a$, can you find a general way to solve for x in terms of a?
12. $x^2 \oplus x \oplus 1 = -1$
13. $x^2 \oplus x \oplus 1 = 0$
14. $x^2 \oplus x \oplus 1 = \frac{1}{2}$
15. $x^2 \oplus x \oplus 1 = 1$
16. $x^2 \oplus x \oplus 1 = 2$
17. For $x^2 \oplus x \oplus 1 = a$, can you find a general way to solve for x in terms of a?
18. For $x \odot (x \oplus 1) = a$, can you find a general way to solve for x in terms of a?

5 Graphing tropical polynomials in one variable

A tropical polynomial is (just as you would expect) any function of the form

$$a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0; \tag{1}$$

here each a_i is any real number and

$$x^{\odot n} = \overbrace{x \odot \cdots \odot x}^{n}.$$

Since it causes no confusion in practice, from now on we will simply write x^n instead of $x^{\odot n}$.

Notice that in order to interpret the expression in (**??**) properly we use the usual conventions that multiplications (i.e. operations involving $\odot$) should be performed before additions (i.e. operations involving $\oplus$).

The main point of this section is to understand what the graph of a polynomial $f(x)$ can look like. We start with the simplest case,

$$l(x) = a \odot x \oplus b;$$

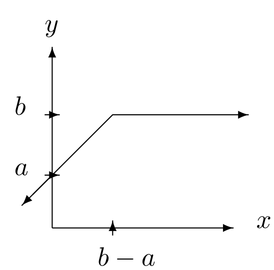
for reasons that should be clear, this is called a tropical line. By definition

$$l(x) = \min\{a + x, b\},$$

or, more explicitly,

$$l(x) = \begin{cases} a + x & \text{if } x \geq b - a \\ b & \text{else.} \end{cases}$$

For the purposes of drawing a graph, let's assume $b > a \geq 0$. Then the graph of $l(x)$ looks like



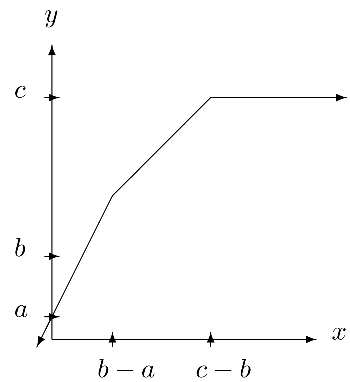
Let's try one more example together. Consider the tropical parabola,

$$p(x) = a \odot x^2 \oplus b \odot x \oplus c.$$

More explicitly, we have

$$p(x) = \min\{a + 2x, bx, c\}.$$

(Make sure you understand this!) Again for the sake of drawing a picture, let's assume $c - b > b - a$. Then we can quickly arrive at the following graph,



The following set of exercises deal with the graphs of general polynomials.

EXERCISES 2

Graph the following expressions

1. $1 \odot x^2 \oplus 2 \odot x \oplus 5$
2. $0 \odot x^2 \oplus 2 \odot x \oplus 5$
3. What happens when $b - a > c - b$? Try graphing
   (a) $-1 \odot x^2 \oplus 2 \odot x \oplus 5$
   (b) $-1 \odot x^2 \oplus 5$

   What does this mean about the uniqueness of graphs of polynomials?
4. Graph the following (for part (c) you may need to simplify the expression using tropical arithmetic first)
   (a) $x \oplus 5$
   (b) $2 \odot x \oplus 1$
   (c) $(x \oplus 5) \odot (2 \odot x \oplus 1)$

   What do these graphs have in common? What do you think this means about the products of tropical polynomials?

6 Roots of polynomials

Recall that in regular algebra, the roots of a polynomial

$$a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0,$$

are the solutions to the equation

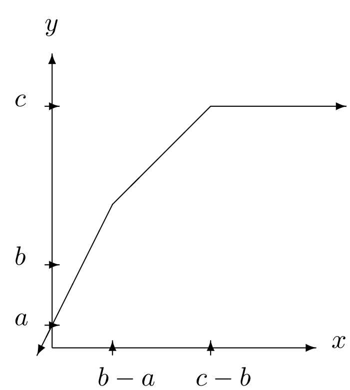
$$a_n \odot x^{\odot n} \oplus a_{n-1} \odot x^{\odot n-1} \oplus \cdots \oplus a_1 \odot x \oplus a_0 = 0.$$

This method of finding roots doesn't work in tropical algebra. Consider the polynomial equation

$$x \oplus -1 = 0.$$

Does this equation have any solutions? No! (Since the minimum of something and -1 must always b less than or equal to -1, and therefore can never be 0).

Instead, let's define the roots of a polynomial to be the breaking points in the graph, or more specifically, the place on the graph where the minimum of all monomials is achieved twice. So, if we consider the quadratic polynomial we used earlier, whose graph looks like



Then the roots are $x = b - a$ and $x = c - b$. Recall that the polynomial was given by

$$a \odot x^2 \oplus b \odot x \oplus c = \min\{a + 2x, b + x, c\}$$

The places where the minimum could be achieved twice are where

$$\begin{aligned} a + 2x = b + x \leq c, &\text{ or} \\ a + 2x = c \leq b + x, &\text{ or} \\ b + x = c \leq a + 2x. & \end{aligned}$$

After solving for x we get

$$x = b - a, \text{ if } 2b - a \leq c, \text{ or}$$
$$x = \frac{1}{2}(c - a), \text{ if } c \leq b + \frac{1}{2}(c - a), \text{ or}$$
$$x = c - b, \text{ if } c \leq a + 2c - 2b.$$

Recall that when we started the problem we decided to assume that $b-a < c-b$ which is the same as $2b - a < c$ and $c < a + 2c - 2b$, but is the opposite of $c < b + \frac{1}{2}(c - a)$. This tells us that the first an third choices should have shown up as roots of our polynomial, but not the second. So, you can find the roots algebraically as well.

EXERCISES 3

Find the roots to all of the polynomials in Exercises 2.

1. $1 \odot x^2 \oplus 2 \odot x \oplus 5$
2. $0 \odot x^2 \oplus 2 \odot x \oplus 5$
3. What happens when $b - a > c - b$?
   (a) $-1 \odot x^2 \oplus 2 \odot x \oplus 5$
   (b) $-1 \odot x^2 \oplus 5$
4. (a) $x \oplus 5$
   (b) $2 \odot x \oplus 1$
   (c) $(x \oplus 5) \odot (2 \odot x \oplus 1)$

Now let's try to find the roots of polynomials with two variables, x, and y. Consider the polynomial

$$a \odot x \oplus b \odot y \oplus c = \min\{a + x, b + y, c\}$$

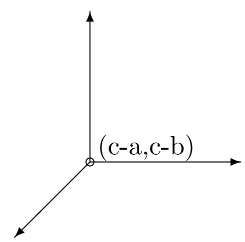
In order to solve this one geometrically, we would need to be able to graph three planes in three dimensional space. That is we would want to look at $z = c$, $z = a + x$ and $z = b + y$. Instead, let's find the roots algebraically. Remember that the roots in this sense are where the minimum of the three monomials is achieved more than once. This can happen in three ways:

$$a + x = b + y \leq c$$
$$a + x = c \leq b + y$$
$$b + y = c \leq a + x$$

After simplifying, this is the union of three rays:

$$y = x + (a - b), \quad x \leq c - a$$
$$x = c - a \quad y \geq c - b$$
$$y = c - b \quad x \geq c - a$$

The picture of this looks like



EXERCISES 4

Find the roots of the following polynomials

1. $5 \odot x \oplus 4 \odot y \oplus 6$
2. $x \odot y \oplus y^2 \oplus 1 \odot y \oplus 0$
3. $x \odot y \oplus -1 \odot y^2 \oplus x \oplus y$
4. $x \odot y \oplus y^2 \oplus x \oplus y$
5. $1 \odot x^2 \oplus x \odot y \oplus 1 \odot y^2 \oplus x \oplus y \oplus 1$