SOLUTIONS TO BRAIN TEASERS

1. The answer is (C). The quick way to see it is to regard the Fibonacci sequence modulo 10 (after all, what you care about is only the unit digits). Then the sequence is

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, ...$$

2. The answer is (B). The 200 terms can be grouped into 100 odd-even pairs, each with a sum of -1. Thus the sum of the first 200 terms is -100, and the average -0.5.

3. The answer is (A). A number congruent to -1 modulo 5 has unit digits 4 or 9. On the other hand being congruent to 1 modulo 4 forces the number to be odd. Hence only the 9 candidates survive. Of these, you can check by hand that only 9, 29, 49, 69 and 89 are equal to 1 mod 4. Among them the primes are 29 and 89.

4. It suffices to look at the polynomial modulo 2. It reduces to

$$P(x) = x^8 + x^5 + x^4 + x^3 + 1.$$

And now it suffices to notice that $P(0) = P(1) = 1$ (modulo 2).

5. The answer is (D). The trick is to notice that v appears in the first row, column and in one diagonal. Hence the sum of the remaining two numbers in each of these lines must be the same. This allows us to determine that $w = 19$ and $x = 22$. But now we know a complete diagonal, hence we know the magic sum has to be 66. It's now easy to determine all the other unknowns.

6. The answer is 8. The number 10^n can be expressed as the product of 2^n and 5^n, neither of which contains the digit 0 up to $n = 7$. $5^8 = 380625$ and all other pairs of positive integers whose product is 10^8 contain a 0 in the units position.

7. The answer is (D). The only two-digit palindrome is 11, hence the three digit palindrome must be less than $\frac{2000}{11} = 181.81818181....$ The only three-digit prime palindromes in this range are $101, 131, 151$ and 181. And finally you need to check that the product of 11 with each of those numbers produces a palindrome as a result.

SOLUTIONS TO BRAIN SQUEEZERS

1. The solution is (D). We notice that the rule to assign the sequence is simply: double the previous number, then look at the fractional part. Hence x_5 is just the fractional part of $32x_0$. What we are asked is that $32x_0 - x_0 = 31x_0$ be an integer. But this happens only if I choose x_0 as a nonnegative integral multiple of $1/31$, and I have precisely 31 such numbers between 0 and 1 (excluding 1).

2. The answer is (B). The slick way to see this is: take the sum and the product of the three relations:

$$S = x + \frac{1}{y} + y + \frac{1}{z} + z + \frac{1}{x} = \frac{22}{3}$$

$$P = (x + \frac{1}{y})(y + \frac{1}{z})(z + \frac{1}{x}) = \frac{28}{3}$$

But now notice that

$$P = xyz + x + y + z + \frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{1}{xyz} = xyz + S + \frac{1}{xyz}$$

Hence

$$xyz + \frac{1}{xyz} = 2$$

Or, after a very little bit of algebra,

$$(xyz - 1)^2 = 0.$$

3. The answer is (B). Note that

$$C = A log_{200} 5 + B log_{200} 2 = log_{200} 5^A + log_{200} 2^B = log_{200}(5^A * 2^B),$$

so $200^C = 5^A * 2^B$. Therefore $5^A * 2^B = 200^C = (5^2 * 2^3)^C = 5^{2C} * 2^{3C}$. By uniqueness of prime factorization, we get $A = 2C$, $B = 3C$, from which it follows, since A, B and C have no common factor, that $C = 1, B = 3, A = 2$.

4. Set $n^2 - 19n + 99 = m^2$, for some positive integer m. Then multiply everything by 4, complete the square on the right hand side, and rearrange things so as to get the difference of two squares on one side:

$$4m^2 - (2n - 19)^2 = 35$$

$$(2m + 2n - 19)(2m - 2n + 19) = 35$$

Now, since the sum of the two factors is $4m$, a positive integer, it follows that the pair $(2m + 2n - 19), (2m - 2n + 19)$, can only be $(1, 35), (5, 7), (7, 5)$ or $(35, 1)$. Now subtracting the second factor from the first and doing a little algebra we discover that n can be only 1,9, 10 or 18. Hence the solution is 38.

5. The answer is (A). If $x > 0$, then the line $x + y = 3$ and the parabola $y + x^2 = 0$ do not intersect. Hence x must be a negative number. In this case, if such a pair of numbers exists at all, by the first condition $y - x = 3$, and hence $x - y = -3$. To determine that there is actually a solution to the system it suffices again to observe that the line and the parabola determined by the equations with $-x$ substituted for the absolute value of x this time do intersect.

6. The answer is (B). If we rename $p = log_y x$, then let's observe that $log_x y = 1/p$. Hence now it amounts to solve a quadratic equation to find that p can be 3 or $1/3$. (And the whole situation is symmetric in x and y so we can just consider one case). So if we set $p = log_y x = 3$ we have that $x = y^3$. Now multiply this equation by y and then use the second relation to deduce that $144 = y^4$, hence $y = \sqrt{12}$. And from here on it's straightforward computations to the result.

SOLUTIONS TO BRAIN SQUISHERS

1. Let us first of all observe our expression modulo 2. We deduce that either all three numbers must be even, or I need to have one even number and two odd numbers. We rule out the first possibility because otherwise we can factor a 16 in the left hand side. But, obviously, 16 does not divide 24. Now let's play around with our expression:

$$x^4 + y^4 + z^4 - 2y^2 z^2 - 2z^2 x^2 - 2x^2 y^2 =$$

$$= (x^2 + y^2 - z^2)^2 - 4x^2y^2 =$$
$$= (x^2 + y^2 - z^2 + 2xy)(x^2 + y^2 - z^2 - 2xy) =$$
$$= ((x+y)^2 - z^2)((x-y)^2 - z^2) =$$
$$= (x+y+z)(x+y-z)(x-y+z)(x-y-z).$$

Since all these factors are even (from the assumption that two of the three numbers must be odd and the third even) and from unique factorization of primes we deduce that the above product can never be equal to 24.

2. Let me call I the integral we are going to evaluate. We notice, first of all, that if we do the substitution $y = a - x$, we obtain

$$I = \int_0^a \frac{f(y-a)}{f(y) + f(a-y)} dy.$$

Now observe that, by adding and subtracting to my numerator the quantity $f(a-x)$, I can also regard

$$I = \int_0^a (1 - \frac{f(x-a)}{f(x) + f(a-x)}) dx = a - I.$$

From which it follows that $I = \frac{a}{2}$.