An Introduction to Non-Cooperative Game Theory
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Introduction

Last week I gave an introduction to modeling interactions that involves conflict and uncertainty in opponent action. You might recall that the model included a thorough description of the players involved, the actions available to each player, and the payoffs each player receives from each action profile. This information is expressed in the form of a payoff matrix whenever possible.

Definition 1: **Players** are interacting parties that have the ability to display various actions during an interaction

Definition 2: An **action** is a particular physical response which evokes consequences, called **payoffs**, for each player involved.

Definition 3: An **action profile** is a characterization of the actions that each player has chosen.

We ended last week by discussing uncertainty, probability, and modeling action choice. In fact, I ended by presenting a formal definition for a strategy. Today, I will briefly cover strategies and average payoffs. Then, we will focus our attention on finding solutions to various games. Our work today should be fun and enlightening!

Outline for October 12, 2005

Section 2.1: Strategies and Average Payoffs
Section 2.2: Best Replies
Section 2.3: Nash Equilibrium

Section 2.1: Strategies and Average Payoffs

A **strategy** is a mathematical description of how a player chooses an action for his/her action set. In fact, we use probability density functions for this. You might recall that a probability density function assigns a probability to each outcome of an experiment. Here, the probability density function shall assign a probability to each action in an action set. Futhermore, the assigned probability will provide us with an idea of the likelihood that a particular action will be chosen by a player. To aid in our understanding of this concept, let's revisit the Prisoner's Dilemma and think about strategies each player can adopt.

The Prisoner's Dilemma

Two suspects are being held separately in prison. They are accused of having conspired together to commit a crime that they have previously agreed together to deny if summoned by the police for questioning. A detective is trying relentlessly to persuade each of them to implicate the other with the following inducement. If neither of them confesses they will both be set free because there is not enough evidence against them; if both confess, they will both be punished; but if only one confesses, the confessor is set free and in addition receives a reward while his partner is punished more severely than if he had confessed.

If we were to write a strategy for player 1 (suspect 1) and player 2 (suspect 2) explicitly, each strategy would be a piecewise function that maps elements of the action set $\{H, I\}$ to the probability that player 1 or player 2 will choose them respectively. These functions are provided below.

$$pdf_1(x) = \begin{cases} x_{11} & if\ x\ =\ Honor \\ x_{12} & if\ x\ =\ Ignore \\ 0 & otherwise \end{cases}$$

subject to the conditions $0 \leq x_{11} \leq 1$, $0 \leq x_{12} \leq 1$, and $x_{11} + x_{12} = 1$

AND

$$pdf_2(x) = \begin{cases} x_{21} & if\ x\ =\ Honor \\ x_{22} & if\ x\ =\ Ignore \\ 0 & otherwise \end{cases}$$

subject to the conditions $0 \leq x_{21} \leq 1$, $0 \leq x_{22} \leq 1$, and $x_{21} + x_{22} = 1$

Example 2.1.1 Examples of strategies available to player 1 of the Prisoner's Dilemma include

$$pdf_1(x) = \begin{cases} \frac{1}{5} & if\ x\ =\ Honor \\ \frac{4}{5} & if\ x\ =\ Ignore \\ 0 & otherwise \end{cases} \quad \text{and} \quad pdf_1(x) = \begin{cases} 1 & if\ x\ =\ Honor \\ 0 & if\ x\ =\ Ignore \\ 0 & otherwise \end{cases}.$$

There are two types of strategies which require discussion: pure and mixed strategies. A **pure strategy** is a strategy in which all probability is assigned to exactly one action. This strategy is predictable since a player who adopts this strategy always utilizes the same action. All other strategies are called **mixed strategies**. These strategies necessarily have non-zero probability assigned to two or more actions. You should think of these strategies as a mixing of two or more actions which ultimately results in unpredictability in action choice.

If each player adopts a strategy which she will use to play the game, we must come up with a creative way to compute payoffs since more than one action profile may arise in any given game. The trick is to calculate the average payoff each player shall receive from a game played according to these strategies. Thankfully, this calculation is not obnoxiously difficult. It requires calculating a weighted average of the payoffs associated with each action profile.

Definition 2.1.1 Average Payoff for Player i

Let a represent any action profile and A be the collection of all action profiles. The average payoff that player i shall receive from an interaction is given by $u_i = \sum_{a \in A} Pr(a) * \pi_i(a)$ where $Pr(a)$ represents the probability of the action profile a. $Pr(a)$ is determined from the strategies $pdf_1, pdf_2, \ldots, pdf_n$.

Did I just lie to you? I said that the calculation was not "obnoxiously difficult." You might disagree with me after glancing at the definition. An example might convince you that the above notation makes the idea look significantly more complicated than it actually is.

Example 2.1.2 (Prisoner's Dilemma continued) Calculate the average payoff that player 1 receives given the strategies

$$pdf_1(x) = \begin{cases} \frac{1}{4} & if\ x\ =\ H \\ \frac{3}{4} & if\ x\ =\ I \\ 0 & otherwise \end{cases} \quad \text{and} \quad pdf_2(x) = \begin{cases} \frac{5}{8} & if\ x\ =\ H \\ \frac{3}{8} & if\ x\ =\ I \\ 0 & otherwise \end{cases},$$

and the payoff matrix below.

$$\begin{array}{ccc} & & \text{PLAYER 2} \\ & & \begin{array}{cc} \text{HONOR} & \text{IGNORE} \end{array} \\ \text{PLAYER 1} & \begin{array}{c} \text{HONOR} \\ \text{IGNORE} \end{array} & \begin{pmatrix} (0,0) & (-4,2) \\ (2,-4) & (-1,-1) \end{pmatrix} \end{array}$$

Recall that the action profile space for this game is given by

$$A = \{(H,H),(H,I),(I,H),(I,I)\}$$

By definition, the average payoff player 1 shall receive is given by

$$\sum_{a \in A} Pr(a) * \pi_1(a) \;=\; Pr((H,H)) * \pi_1((H,H)) + Pr((H,I)) * \pi_1((H,I)) + Pr((I,H)) * \pi_1((I,H)) + Pr((I,I)) * \pi_1((I,I)).$$

Assuming each player makes her choice independent of all other players, we arrive at the following result:

$$\begin{aligned} \sum_{a \in A} Pr(a) * \pi_1(a) &= pdf_1(H) * pdf_2(H) * \pi_1((H,H)) + \ldots + pdf_1(I) * pdf_2(I) * \pi_1((I,I)) \\ &= (\tfrac{1}{4} * \tfrac{5}{8} * 0) \;+\; (\tfrac{1}{4} * \tfrac{3}{8} * -4) \;+\; (\tfrac{3}{4} * \tfrac{5}{8} * 2) \;+\; (\tfrac{3}{4} * \tfrac{3}{8} * -1) = \tfrac{9}{32}. \end{aligned}$$

The above value is the average payoff that player 1 receives from this interaction given the above strategies and payoff matrix. The specific value $u_1 = \frac{9}{32}$ indicates that the average result for player 1 under these conditions is no jail time (sweet!) coupled with a minimal reward.

Section 2.1 Problems

This problem set will serve as continued analysis of the Prisoner's Dilemma game.

Problem 1 Calculate the average payoff that player 2 receives given the strategies and payoff matrix defined in Example 2.1.2.

Problem 2 What does your solution from Problem 1 literally mean in the context of the game?

Problem 3 Using your results from Example 2.1.2 and Problem 1, which player is least satisfied with her average payoff and will most likely want to switch strategies?

Problem 4 Fix the strategy pdf_1 for player 1 and find another strategy for player 2 that provides a greater average payoff to player 2 than the strategy pdf_2 does.

Problem 5 Can you find a strategy for player 2 which provides a maximum average payoff to player 2 given that player 1 uses the strategy pdf_1? Label this strategy for player 2 $\overrightarrow{pdf}_2$.

Problem 6 Can you find a strategy for player 1 which provides a maximum average payoff to player 1 given that player 2 uses the strategy $\overrightarrow{pdf}_2$?

Section 2.2: Best Replies

Problems 4, 5, and 6 from the Section 2.1 problem set illustrate a couple of important ideas in non-cooperative game theory. First, I led you to believe that each player should make strategy decisions based on the average payoff they received from these strategies. Second, these decisions should be based on a maximization criterion. These two ideas are in fact fundamental assumptions about player behavior and are best summed up by the best reply concept.

Definition 2.2.1 Best Reply

Assume there are n players in our game and that $n-1$ of them have chosen some strategy. Futhermore, assume that the n^{th} player can cycle through her strategies and compare the average payoffs received from each of them. A best reply to the current state of the field ($n-1$ players each fixed on some strategy) is a strategy that maximizes the average payoff that the n^{th} player receives.

Example 2.2.1 Look back at Section 2.1 problem 5. Hopefully you were able to conclude that

$$pdf_2(x) = \begin{cases} 0 & if\ x = H \\ 1 & if\ x = I \\ 0 & otherwise \end{cases} \quad \text{was player 2's best reply to} \quad pdf_1(x) = \begin{cases} \frac{1}{4} & if\ x = H \\ \frac{3}{4} & if\ x = I \\ 0 & otherwise \end{cases}.$$

In the next problem (Section 2.1 problem 6), you should have arrived at the conclusion that

$$pdf_1(x) = \begin{cases} 0 & if\ x = H \\ 1 & if\ x = I \\ 0 & otherwise \end{cases} \quad \text{was player 1's best reply to} \quad pdf_2(x) = \begin{cases} 0 & if\ x = H \\ 1 & if\ x = I \\ 0 & otherwise \end{cases}.$$

It turns out that finding best replies is vital to finding solutions for a game. To this end, we should practice finding best replies. Instead of using a guess and check method or a more detailed maximization approach, there is a relatively simple method that we can use to find best replies. This method reduces the work necessary to find best replies and does not require any inequality algebra!

Method for finding Best Replies

Assume that there are n players in our game and that $n-1$ of them have chosen some strategy. To find the best reply for the n^{th} player,

1. Write each pure strategy available to the n^{th} player.
2. Compute the average payoff for each of these pure strategies against the field.
3. Determine which pure strategies yield a maximum average payoff.
4. Best replies are all available strategies which encorporate these pure strategies only.

Section 2.2 Problems

Use the payoff matrix below to answer the following questions.

$$\begin{array}{cc} & \text{PLAYER 2} \\ & \begin{array}{cc} \quad A & \quad\quad B \end{array} \\ \text{PLAYER 1} \begin{array}{c} A \\ B \end{array} & \begin{pmatrix} (1,-1) & (-1,1) \\ (-1,1) & (1,-1) \end{pmatrix} \end{array}$$

Problem 1 Show player 1 has a unique best reply to

$$pdf_2(x) = \begin{cases} \frac{1}{3} & if\ x\ =\ A \\ \frac{2}{3} & if\ x\ =\ B \\ 0 & otherwise \end{cases}.$$

Problem 2 Find all best replies for player 1 to

$$pdf_2(x) = \begin{cases} \frac{1}{2} & if\ x\ =\ A \\ \frac{1}{2} & if\ x\ =\ B \\ 0 & otherwise \end{cases}.$$

Section 2.3: Nash Equilibrium

Let's return our attention to the Prisoner's Dilemma again. So far we have shown that if our players initially choose the strategies

$$pdf_1(x) = \begin{cases} \frac{1}{4} & if\ x\ =\ H \\ \frac{3}{4} & if\ x\ =\ I \\ 0 & otherwise \end{cases} \quad \text{and} \quad pdf_2(x) = \begin{cases} \frac{5}{8} & if\ x\ =\ H \\ \frac{3}{8} & if\ x\ =\ I \\ 0 & otherwise \end{cases},$$

and the payoff matrix is given by

$$\begin{array}{cc} & \text{PLAYER 2} \\ & \begin{array}{cc} \text{HONOR} & \text{IGNORE} \end{array} \\ \text{PLAYER 1} \quad \begin{array}{c} \text{HONOR} \\ \text{IGNORE} \end{array} & \begin{pmatrix} (0,0) & (-4,2) \\ (2,-4) & (-1,-1) \end{pmatrix} \end{array},$$

then player 2 would like to switch from using pdf_1 to using the Ignore pure strategy. This results in player 1 wanting to switch to the Ignore pure strategy as well. If we would have continued this analysis we would have determined that

$$pdf_1(x) = \begin{cases} 0 & if\ x\ =\ H \\ 1 & if\ x\ =\ I \\ 0 & otherwise \end{cases} \quad \text{was player 1's best reply to} \quad pdf_2(x) = \begin{cases} 0 & if\ x\ =\ H \\ 1 & if\ x\ =\ I \\ 0 & otherwise \end{cases}$$

AND

$$pdf_2(x) = \begin{cases} 0 & if\ x\ =\ H \\ 1 & if\ x\ =\ I \\ 0 & otherwise \end{cases} \quad \text{was player 2's best reply to} \quad pdf_1(x) = \begin{cases} 0 & if\ x\ =\ H \\ 1 & if\ x\ =\ I \\ 0 & otherwise \end{cases}.$$

That is, neither contestant wants to switch strategies when each is using the Ignore pure strategy. Such a state where no one wants to switch strategies is called a **Nash equilibrium**. A Nash equilibrium is the solution concept that we will adopt.

Definition 2.3.1 Nash Equilibrium

A Nash equilibrium of a game is a collection of strategies (one for each player involved) in which each player's strategy is a best reply to the current state of the field.

Nash equilibrium for 2-Player games (where strategies for player 1 and player 2 are represented by pdf_1 and pdf_2 respectively) are all pairs of strategies (pdf_1, pdf_2) satisfying the following two conditions simultaneously:

1. pdf_1 is a best reply to pdf_2
2. pdf_2 is a best reply to pdf_1.

Example 2.3.1 Nash equilibrium of the Prisoner's Dilemma game.

A Nash equilibrium of the Prisoner's Dilemma game is

$$\left(pdf_1(x) = \begin{cases} 0 & if\ x\ =\ H \\ 1 & if\ x\ =\ I \\ 0 & otherwise \end{cases},\quad pdf_2(x) = \begin{cases} 0 & if\ x\ =\ H \\ 1 & if\ x\ =\ I \\ 0 & otherwise \end{cases} \right)$$

since we have successfully shown that pdf_1 is a best reply to pdf_2 AND pdf_2 is a best reply to pdf_1. It turns out that this is the only Nash equilibrium for this game. When this is the case, we call the Nash equilibrium a **strict Nash equilibrium**.

Section 2.3 Problems:

Problem 1 Matching Pennies Game

Two people choose, simultaneously, whether to show the head or tail of a coin. If they show the same side, person 2 pays person 1 a dollar; if they show different sides, person 1 pays person 2 a dollar. Each person cares only about the amount of money she receives, and naturally prefers to receive more than less. A reasonable payoff matrix for this game is provided for you below.

$$\begin{array}{ccc} & & \text{PLAYER 2} \\ & & \begin{array}{cc} \text{HEADS} & \text{TAILS} \end{array} \\ \text{PLAYER 1} & \begin{array}{c} \text{HEADS} \\ \text{TAILS} \end{array} & \begin{pmatrix} (1,-1) & (-1,1) \\ (-1,1) & (1,-1) \end{pmatrix} \end{array}$$

Show that the pair of strategies

$$\left(pdf_1(x) = \begin{cases} \frac{1}{2} & if\ x\ =\ H \\ \frac{1}{2} & if\ x\ =\ T \\ 0 & otherwise \end{cases},\quad pdf_2(x) = \begin{cases} \frac{1}{2} & if\ x\ =\ H \\ \frac{1}{2} & if\ x\ =\ T \\ 0 & otherwise \end{cases} \right)$$

is a Nash equilibrium for this game.

Problem 2 Battle of the Friends game

Adam and Bob want to catch a concert together this weekend. After a little research online, they discovered that two concerts still have seats available: one by 50-Cent and one by George Winston. Bob prefers to see 50-Cent while Adam prefers to see George Winston. If they go to different concerts, each of them is equally unhappy listening to the music of the other artist alone. A reasonable payoff matrix is given below.

$$\begin{array}{ccc} & & \text{BOB} \\ & & \begin{array}{cc} 50 & GW \end{array} \\ \text{ADAM} & \begin{array}{c} 50 \\ GW \end{array} & \begin{pmatrix} (8,10) & (2,3) \\ (3,2) & (10,8) \end{pmatrix} \end{array}$$

ADAM = PLAYER 1
BOB = PLAYER 2

Show that the both pairs of strategies

$$\left(pdf_1(x) = \begin{cases} 0 & if\ x\ =\ 50 \\ 1 & if\ x\ =\ GW \\ 0 & otherwise \end{cases},\quad pdf_2(x) = \begin{cases} 0 & if\ x\ =\ 50 \\ 1 & if\ x\ =\ GW \\ 0 & otherwise \end{cases} \right)$$

AND

$$\left(pdf_1(x) = \begin{cases} 1 & if\ x\ =\ 50 \\ 0 & if\ x\ =\ GW \\ 0 & otherwise \end{cases},\quad pdf_2(x) = \begin{cases} 1 & if\ x\ =\ 50 \\ 0 & if\ x\ =\ GW \\ 0 & otherwise \end{cases} \right)$$

are a Nash equilibria for this game.

Problem 3 Paper-Rock-Scissors game

This is a children's game known as the Paper-Rock-Scissors game: rock beats scissors, scissors beats paper, and paper beats rock. Consider the following payoff matrix

PLAYER 2

$$\begin{array}{cc}
 & \begin{array}{ccc} \quad R & \quad\; S & \quad\; P \end{array} \\
\text{PLAYER 1} \begin{array}{c} R \\ S \\ P \end{array} & \left(\begin{array}{ccc} (1,1) & (2,0) & (0,2) \\ (0,2) & (1,1) & (2,0) \\ (2,0) & (0,2) & (1,1) \end{array} \right).
\end{array}$$

What are the Nash equilibria of this game?

Problem 4 Try to find the Nash equilibrium of a 2-player game which has the following payoff matrix:

$$\begin{array}{ccc} & & \text{Player 2} \\ & & \begin{array}{cc} H & D \end{array} \\ \text{Player 1} & \begin{array}{c} H \\ D \end{array} & \begin{pmatrix} (2,2) & (6,0) \\ (0,6) & (3,3) \end{pmatrix} \end{array}$$

Hint: Try Nash equilibrium that are composed of pure strategies only first. If all these possibilities fail, then the Nash equilibrium must be a mixed strategy.

Problem 5 Try to find the Nash equilibrium of a 2-player game which has the following payoff matrix:

$$\begin{array}{ccc} & & \text{PLAYER 2} \\ & & \begin{array}{cc} \text{H} & \text{D} \end{array} \\ \text{PLAYER 1} & \begin{array}{c} \text{H} \\ \text{D} \end{array} & \begin{pmatrix} (-2,-2) & (2,0) \\ (0,2) & (1,1) \end{pmatrix} \end{array}$$

Use the same hint I gave in the previous problem!