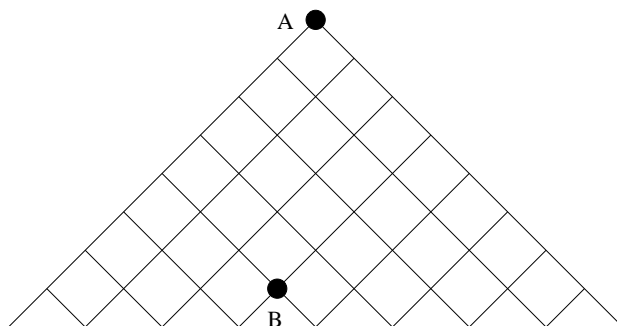


Counting, Permutations, and Probability

Consider the following problems:

1. How many different minimal length paths are there to from point A to B in the graph below?



2. Suppose the list of pizza toppings is pepperoni, sausage, green peppers, onions, mushrooms, and alfalfa sprouts. How many different two topping pizzas are there?
3. Suppose Chuck, Sally, Linus, Lucy, Snoopy, Marcy, and Peppermint Patty are going to sit at a round table to play a game. How many different seating arrangements are there?

Each of these problems can be solved by simply listing every possible configurations and then counting the number of desired configurations. The goal of this session is to practice efficient counting techniques.

Here are four fundamentally different questions:

1. Suppose I have n distinct letters in my 'alphabet'. How many different 'words' of length k are there?

2. Suppose I have n distinct letters. How many different words of length k are there whose letters are all different?
3. Suppose I have have n distinct letters. How many collections of k distinct letters are there? (Assume $0 \leq k \leq n$.)
4. Suppose there are n distinct letters. How many collections of k letters (with some possibly repeated) are there?

Here are two basic counting principles to keep in mind:

Addition Principle. If a collection C of objects can be divided into two collections S and T which have no objects in common, then

$$\#C = \#S + \#T.$$

Multiplication Principle. If a collection C can be divided into m types and each type can be broken up into k subtypes, then

$$\#C = mk.$$

Here are some relevant mathematical definitions:

Permutations. The number of ways to select an ordered arrangement (a permutation) of k objects from a collection of n objects is

$$P(n, k) = n(n-1) \dots (n-k+1) = \frac{n!}{(n-k)!}.$$

Combinations. The number of ways to select k objects (unordered) from a collection of n objects is

$$C(n, k) = \frac{n(n-1) \dots (n-k+1)}{k!} = \binom{n}{k}.$$

Arrangements with repetitions. If there are n objects, r_1 of type one, r_2 of type two, \dots r_m of type m , so that $n = r_1 + \dots + r_m$, then there the number of ordered arrangements is

$$P(n; r_1, \dots, r_m) = \frac{n!}{r_1! \dots r_m!} = \binom{n}{r_1} \binom{n-r_1}{r_2} \dots \binom{n-r_1-\dots-r_{m-1}}{r_m}.$$

Selections with repetitions. The number of ways to select r objects (unordered) with possible repetition from a collection of n types of objects is

$$C(n+r-1, r) = C(n+r-1, n-1).$$

If x_i equals the number of objects of type i chosen, we want to count the number of ways to write $r = x_1 + \dots + x_n$, where each $x_i \geq 0$.



Above we see a partition of the number 8 into five non-negative integers:

$$8 = 2 + 1 + 0 + 3 + 2.$$

Problems.

1. How many ways are there to arrange the cards in a 52 card deck?
2. How many different ways are there to give grades (A, B, C, D, or E) to a class of 20 students?
3. How many ways are there to seat five girls and five boys around a table? What if you are asked to seat them boy-girl-boy-girl, etc.?
4. If a coin is flipped 10 times, what is the probability of getting 8 or more heads? What is the probability of getting exactly 8 heads in a row? At least 8 heads in a row?
5. If there are six different French books, eight different Russian books, and five different Vulcan books, how many ways are there to arrange the books on a shelf with all books of the same language grouped together?
6. What is the probability that a 4-digit telephone extension has no repeated digits?
7. What is the probability that two or more people in a random group of 25 people have a the same birthday (mm/dd)?
8. How many ways are there to roll a die six times and obtain a sequence of outcomes with one '1', three '5' 's, and two '6' 's?
9. If three identical dice are rolled, how many different outcomes can be recorded? What are their probabilities?
10. Suppose a toy company makes cubical blocks out of wood and paints the each side of the cube with either RED or BLUE. How many different types of blocks can the toy company make?

11. How many secret codes are possible if the first guess is RBYG and the score is O?
12. the score is OX?
13. the score is OOX?
14. The first guess is RRWY and the score is O?
15. the score is OX?
16. the score is OO?
17. What is the code if GYRB gives a score OOOO and the code GBRY gives the score OOXX?
18. Find a fourth guess which will let you figure out the code if GYPR gives XX, YBGO gives OX and BOYY gives O.