Angling for an n-gon

It often happens in mathematics that you can solve a wide variety of problems using a few elementary ideas and a bit (perhaps a lot!) of work. Contest problems provide a striking example of this principle, as we hope to convince you over the next few weeks. We will begin with geometric problems in the plane, in which triangles, polygons, and circles play prominent roles.

To get to know these main characters, let's begin with the triangle, a simple polygon consisting of 3 vertices and the segments which join them. (To have a legitimate triangle, we assume that the 3 vertices don't lie on the same line.) Triangles come in a variety of shapes and sizes - equilateral, isosceles, scalene, right, acute, obtuse, etc. You will be amazed at how often particular triangles appear in contest problems: the equilateral triangle, in which all angles equal 60° and all sides are equal; its bisection, the 30-60-90 (1-2-$\sqrt{3}$) triangle; the isosceles 45-45-90 (1-1-$\sqrt{2}$) right triangle; other right triangles, including favorites like the 3-4-5 and 5-12-13 triangles.

One universal property of *all* triangles is that the sum of the three interior angles is 180°. You can decompose n-gons into triangles to compute the sum of their interior angles.

Problem 1. Prove that the interior angles of a triangle always add up to 180°.

Problem 2. Prove that the sum of the interior angles of a convex n-gon is 180(n-2). (Note that this formula also holds for nonconvex polygons.)

Problem 3. Suppose that the angles of a pentagon form an arithmetic progression with a common difference of 20°. What is the average of its interior angles?

Size matters...

Now that we know all about the interior angles of a convex n-gon, let's figure out how to calculate the areas of such figures. The easiest area to compute is that of a rectangle; by imposing a b-by-h grid on the rectangle ABCD in Figure 1, we see that its area must be the product of its base and height. The area won't change if we slide the top edge CD to the left or right by some distance, as long as it remains parallel to AB. Thus, the area of a parallelogram is also the product of its base and height.

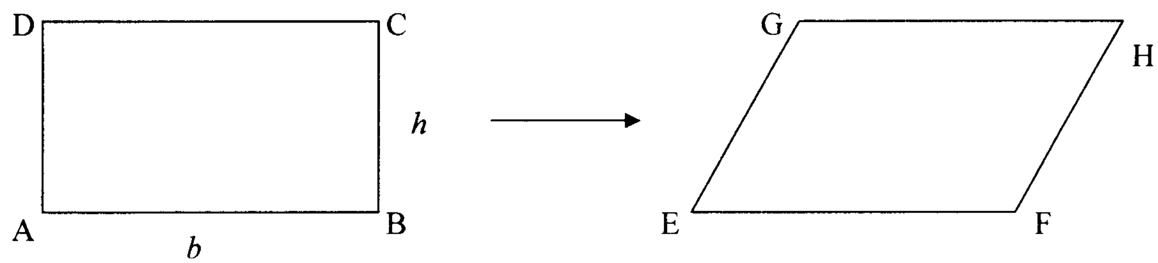


Figure 1

Given a triangle, we can inscribe it in a rectangle as shown in Figure 2. Since triangles ADC and BEF are congruent to triangles AFC and BFC, respectively, we see that the area of triangle ABC is half the area of rectangle ABED. In other words, Area(ΔABC) = ½ *base*height.

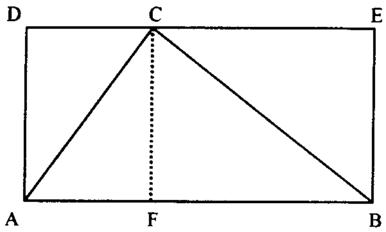
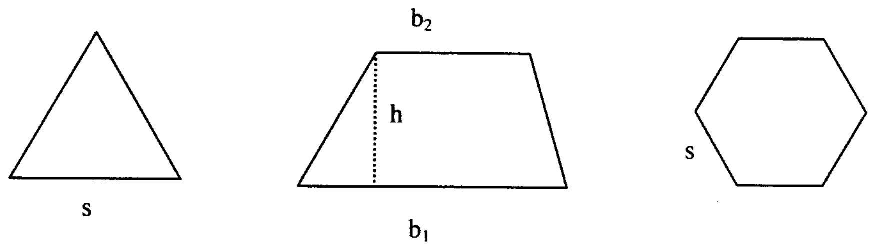


Figure 2

We can use these two area formulas to calculate the areas of many different regions, especially when we combine them with the Pythagorean Theorem. Try to prove the following area formulas for an equilateral triangle, a trapezoid, and a regular hexagon; we emphasize that the basic ideas behind these formulas are much more important than the formulas themselves!



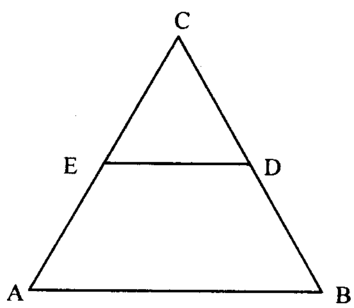
Area = $\dfrac{s^2\sqrt{3}}{4}$

Area = $\dfrac{(b_1 + b_2)h}{2}$

Area = $\dfrac{3s^2\sqrt{3}}{2}$

<u>...but every n-gon looks small from far away</u>

Another technique that can be used in many different geometry problems is the method of similar triangles, which boils down to the fact that two triangles with the same angles are scale models of one another (Figure 3). The challenge in many problems is to *find* the similar triangles in a given figure in order to take advantage of them.



DE is parallel to AB, so Δ ABC and ΔEDC are similar. Thus,

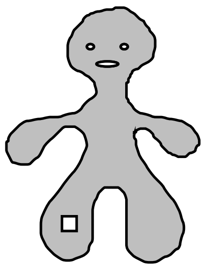
$$\frac{AB}{ED} = \frac{AC}{EC} = \frac{BC}{DC}.$$

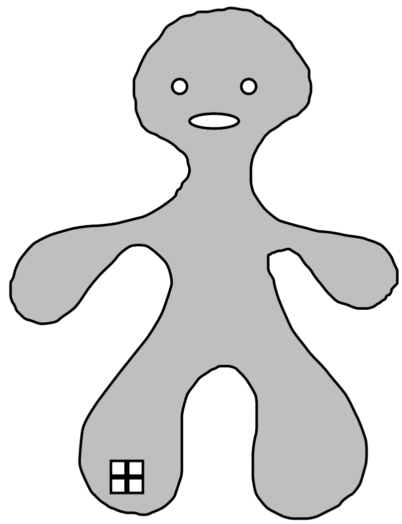
Figure 3

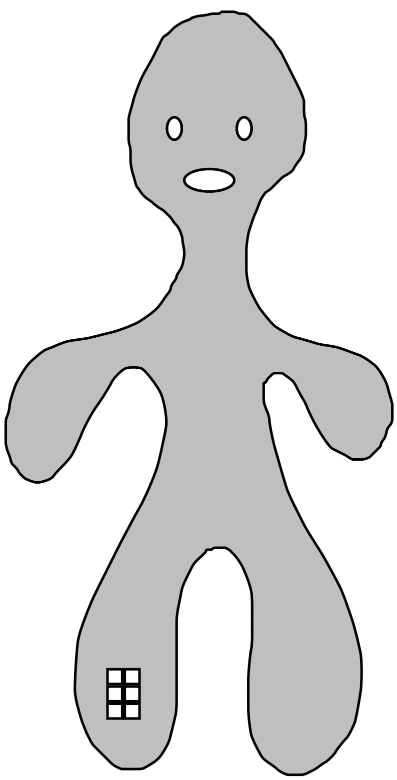
Bob Transforms Himself

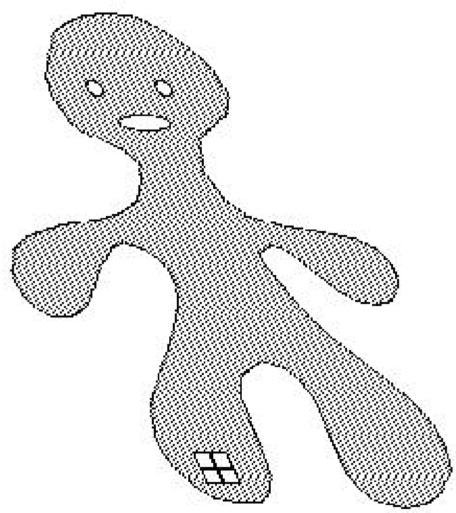
This is Bob: The bigger Bob copies have been rescaled, as indicated by what happened to the little square in his right leg.

How many times bigger are the rescaled Bobs' areas and perimeters, in cases where you can say?



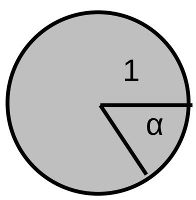






The length of a polygon (or even a curve) can be thought of as the sum of the lengths of (possibly many small) disjoint line segments which make it up. So if you rescale the plane or space by a uniform factor (the same in each direction), each of the line segments on the perimeter will be rescaled by this same factor, and so will the entire length. Similarly, the area of a region is the sum of the areas of rectangles which partition it. If you scale the plane uniformly by a factor then each side of the rectangle becomes longer by that factor, and so the area is multiplied by the square of the scaling constant. You can also scale the plane by different amounts in the horizontal and vertical direction, or by shearing transformations. The effects on area are the same as on the rectangles which partition it, but since lengths of line segments will scale according to their slope, you probably can't predict the effect on the total perimeter of a complicated person like Bob. Similar scaling principles apply in higher dimensions.

Scaling properties help you remember formulas for length, area, volume, and can also be helpful in working contest problems.

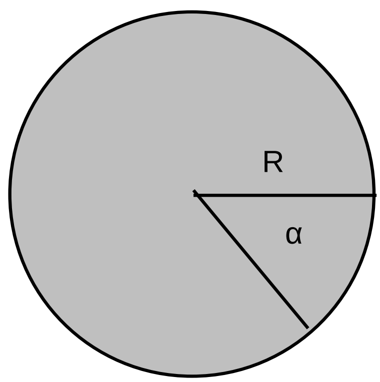


arc length = α radians

total circumference = 2π

sector area = $\alpha/2$

total area = π

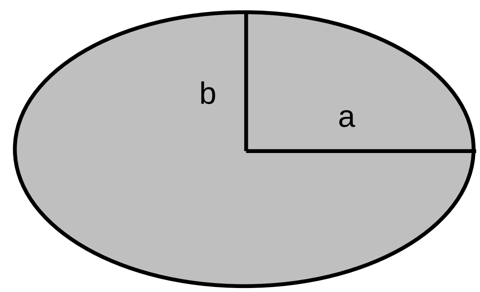


arc length = αR radians

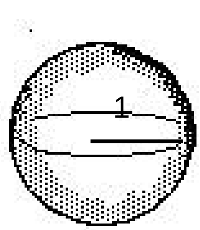
total circumference = $2\pi R$

sector area = $(\alpha/2)R^2$

total area = πR^2

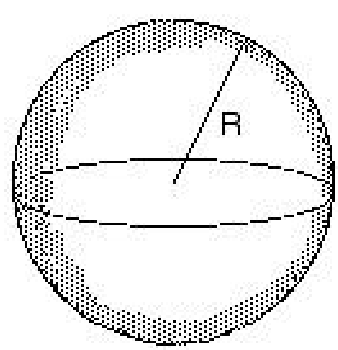


Area = πab



volume = $(4/3)\pi$

surface area = 4π



volume = $(4/3)\pi R^3$

surface area = $4\pi R^2$

APPENDIX II Selected Formulas

1 The sum of the measures of the interior angles of an n-sided convex polygon $= (n - 2)180$.

2 The sum of the measures of the exterior angles of any convex polygon is constant, 360.

3 The area of a rectangle:
$K = bh$.

4a The area of a square:
$K = s^2$.

4b The area of a square:
$K = \frac{1}{2}d^2$.

5a The area of any triangle:
$K = \frac{1}{2}bh$, where b is the base and h is the altitude.

5b The area of any triangle:
$K = \frac{1}{2}ab \sin C$.

5c The area of any triangle:
$K = \sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a + b + c)$.

5d The area of a right triangle:
$K = \frac{1}{2}l_1l_2$, where l is a leg.

5e The area of an equilateral triangle:
$K = \frac{s^2\sqrt{3}}{4}$, where s is any side.

5f The area of an equilateral triangle:
$K = \frac{h^2\sqrt{3}}{3}$, where h is the altitude.

6a The area of a parallelogram:
$K = bh$.

6b The area of a parallelogram:
$K = ab \sin C$.

7 The area of a rhombus:
$K = \frac{1}{2}d_1d_2$.

8 The area of a trapezoid:
$K = \frac{1}{2}h(b_1 + b_2)$.

9 The area of a regular polygon:
$K = \frac{1}{2}ap$, where a is the apothem and p is the perimeter.

10 The area of a circle:
$K = \pi r^2 = \frac{\pi d^2}{4}$, where d is the diameter.

11 The area of a sector of a circle:
$K = \frac{n}{360}\pi r^2$, where n is the measure of the central angle.

12 The circumference of a circle:
$C = 2\pi r$.

13 The length of an arc of a circle:
$L = \frac{n}{360}2\pi r$, where n is the measure of the central angle of the arc.

Basic Identities

$\tan t = \dfrac{\sin t}{\cos t}$ $\qquad$ $\cot t = \dfrac{\cos t}{\sin t}$ $\qquad$ $\cot t = \dfrac{1}{\tan t}$

$\sec t = \dfrac{1}{\cos t}$ $\qquad$ $\csc t = \dfrac{1}{\sin t}$ $\qquad$ $\sin^2 t + \cos^2 t = 1$

$1 + \tan^2 t = \sec^2 t$ $\qquad$ $1 + \cot^2 t = \csc^2 t$

Cofunction Identities

$\sin\left(\dfrac{\pi}{2} - t\right) = \cos t$ $\qquad$ $\cos\left(\dfrac{\pi}{2} - t\right) = \sin t$ $\qquad$ $\tan\left(\dfrac{\pi}{2} - t\right) = \cot t$

Odd-even Identities

$\sin(-t) = -\sin t$ $\qquad$ $\cos(-t) = \cos t$ $\qquad$ $\tan(-t) = -\tan t$

Addition Formulas

$\sin(s + t) = \sin s \cos t + \cos s \sin t$ $\qquad$ $\sin(s - t) = \sin s \cos t - \cos s \sin t$

$\cos(s + t) = \cos s \cos t - \sin s \sin t$ $\qquad$ $\cos(s - t) = \cos s \cos t + \sin s \sin t$

$\tan(s + t) = \dfrac{\tan s + \tan t}{1 - \tan s \tan t}$ $\qquad$ $\tan(s - t) = \dfrac{\tan s - \tan t}{1 + \tan s \tan t}$

Double Angle Formulas

$\sin 2t = 2 \sin t \cos t$ $\qquad$ $\tan 2t = \dfrac{2 \tan t}{1 - \tan^2 t}$

$\cos 2t = \cos^2 t - \sin^2 t = 1 - 2\sin^2 t = 2\cos^2 t - 1$

Half Angle Formulas

$\sin\dfrac{t}{2} = \pm\sqrt{\dfrac{1 - \cos t}{2}}$ $\qquad$ $\cos\dfrac{t}{2} = \pm\sqrt{\dfrac{1 + \cos t}{2}}$ $\qquad$ $\tan\dfrac{t}{2} = \dfrac{1 - \cos t}{\sin t}$

Product Formulas

$2 \sin s \cos t = \sin(s + t) + \sin(s - t)$ $\qquad$ $2 \cos s \cos t = \cos(s + t) + \cos(s - t)$

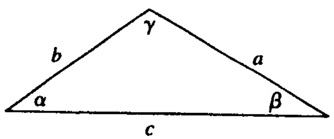
$2 \cos s \sin t = \sin(s + t) - \sin(s - t)$ $\qquad$ $2 \sin s \sin t = \cos(s - t) - \cos(s + t)$

Factoring Formulas

$\sin s + \sin t = 2\cos\dfrac{s - t}{2}\sin\dfrac{s + t}{2}$ $\qquad$ $\cos s + \cos t = 2\cos\dfrac{s + t}{2}\cos\dfrac{s - t}{2}$

$\sin s - \sin t = 2\cos\dfrac{s + t}{2}\sin\dfrac{s - t}{2}$ $\qquad$ $\cos s - \cos t = -2\sin\dfrac{s + t}{2}\sin\dfrac{s - t}{2}$

Laws of Sines and Cosines

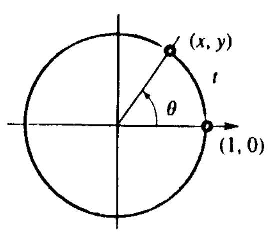


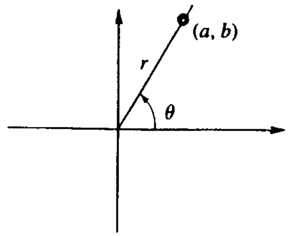
$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b} = \frac{\sin\gamma}{c}$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha$$

Fold here

TRIGONOMETRY

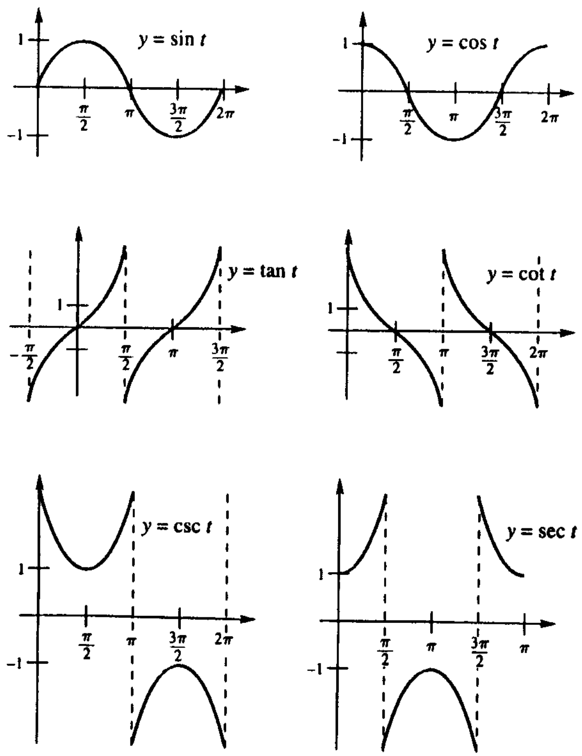




$\sin t = \sin\theta = y = \dfrac{b}{r}$ $\qquad$ $\cos t = \cos\theta = x = \dfrac{a}{r}$

$\tan t = \tan\theta = \dfrac{y}{x} = \dfrac{b}{a}$ $\qquad$ $\cot t = \cot\theta = \dfrac{x}{y} = \dfrac{a}{b}$

Graphs



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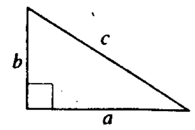
GEOMETRY

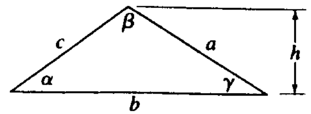


Right triangle

Triangles

Pythagorean Theorem

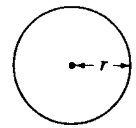
$a^2 + b^2 = c^2$



Any triangle

Angles $\alpha + \beta + \gamma = 180°$

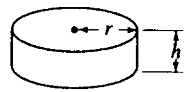
Area $A = \frac{1}{2}bh$



Circles

Circumference $\quad C = 2\pi r$

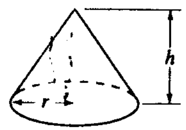
Area $\quad A = \pi r^2$



Cylinders

Surface area $\quad S = 2\pi r^2 + 2\pi rh$

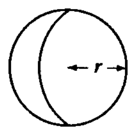
Volume $\quad V = \pi r^2 h$



Cones

Surface area $\quad S = \pi r^2 + \pi r\sqrt{r^2 + h}$

Volume $\quad V = \frac{1}{3}\pi r^2 h$



Spheres

Surface area $\quad S = 4\pi r^2$

Volume $\quad V = \frac{4}{3}\pi r^3$

CONVERSIONS

1 inch = 2.54 centimeters

1 liter = 1000 cubic centimeters

1 kilogram = 2.20 pounds

π radians = 180 degrees

1 kilometer = 0.62 miles

1 liter = 1.057 quarts

1 pound = 453.6 grams

1 cubic foot = 7.48 gallons

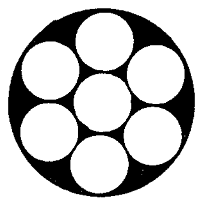
1 The consecutive angles of a trapezoid form an arithmetic sequence. If the smallest angle is 75°, then the largest angle is

(A) 95° (B) 100° (C) 105° (D) 110° (E) 115°

2 Five equilateral triangles, each with side $2\sqrt{3}$, are arranged so they are all on the same side of a line containing one side of each. Along this line, the midpoint of the base of one triangle is a vertex of the next. The area of the region of the plane that is covered by the union of the five triangular regions is

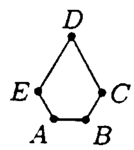
(A) 10 (B) 12 (C) 15 (D) $10\sqrt{3}$ (E) $12\sqrt{3}$

3 Each of the small circles in the figure has radius one. The innermost circle is tangent to the six circles that surround it, and each of those circles is tangent to the large circle and to its small-circle neighbors. Find the area of the shaded region.



(A) π (B) 1.5π (C) 2π (D) 3π (E) 3.5π

4 The convex pentagon $ABCDE$ has $\angle A = \angle B = 120°$, $EA = AB = BC = 2$ and $CD = DE = 4$. What is the area of $ABCDE$?



(A) 10 (B) $7\sqrt{3}$ (C) 15
(D) $9\sqrt{3}$ (E) $12\sqrt{5}$

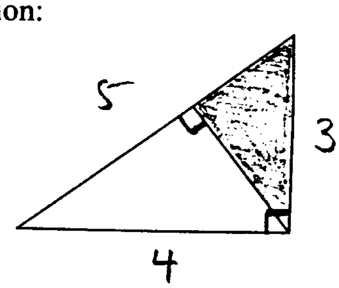
5 Let ΔXOY be a right-angled triangle with $m\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Given that $XN = 19$ and $YM = 22$, find XY.

(A) 24 (B) 26 (C) 28 (D) 30 (E) 32

6 An 8 by $2\sqrt{2}$ rectangle has the same center as a circle of radius 2. The area of the region common to both the rectangle and the circle is

(A) 2π (B) $2\pi + 2$ (C) $4\pi - 4$ (D) $2\pi + 4$
(E) $4\pi - 2$

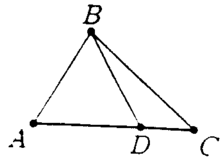
7 Compute the area of the shaded region:



8

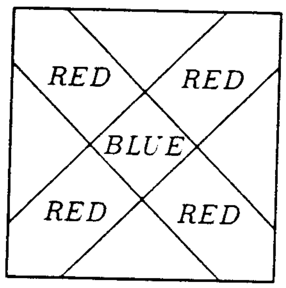
In $\triangle ABC$, $AB = 5$, $BC = 7$, $AC = 9$ and D is on $\overline{AC}$ with $BD = 5$. Find the ratio $AD : DC$.

(A) $4:3$ **(B)** $7:5$ **(C)** $11:6$ **(D)** $13:5$ **(E)** $19:8$



9

A square flag has a red cross of uniform width with a blue square in the center on a white background as shown. (The cross is symmetric with respect to each of the diagonals of the square.) If the entire cross (both the red arms and the blue center) takes up 36% of the area of the flag, what percent of the area of the flag is blue?



(A) .5 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 6

10

Spot's doghouse has a regular hexagonal base that measures one yard on each side. He is tethered to a vertex with a two-yard rope. What is the area, in square yards, of the region outside the doghouse that Spot can reach?

(A) $\frac{2}{3}\pi$ **(B)** 2π **(C)** $\frac{5}{2}\pi$ **(D)** $\frac{8}{3}\pi$ **(E)** 3π

11

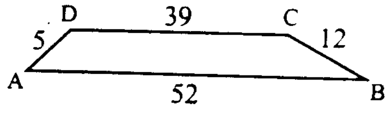
The measures (in degrees) of the interior angles of a convex hexagon form an arithmetic sequence of positive integers. Let m° be the measure of the largest interior angle of the hexagon. The largest possible value of m° is

(A) 165° **(B)** 167° **(C)** 170° **(D)** 175° **(E)** 179°

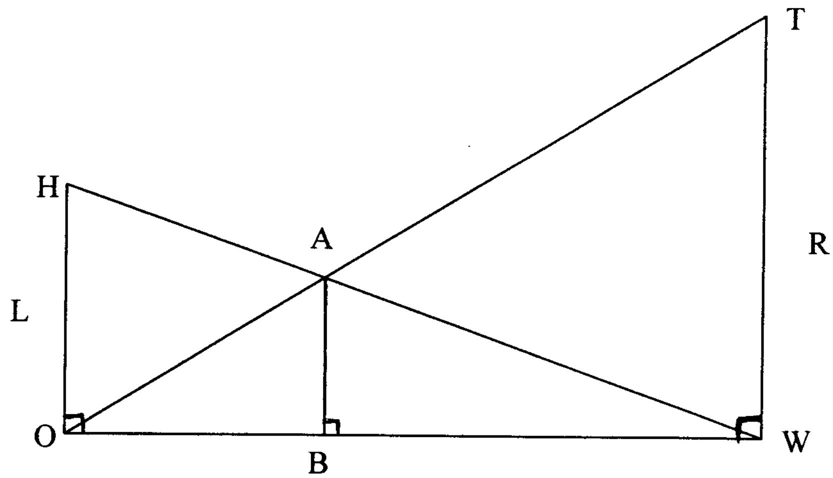
12

In trapezoid $ABCD$ with bases $\overline{AB}$ and $\overline{CD}$, we have $AB = 52$, $BC = 12$, $CD = 39$, and $DA = 5$. The area of $ABCD$ is

(A) 182 **(B)** 195 **(C)** 210 **(D)** 234 **(E)** 260



(13) In the figure below, ΔHOW has a right angle at O, ΔTWO has a right angle at W, and AB is perpendicular to OW. If OH has length L and WT has length R, what is the length of segment AB?



(14) In the triangle shown, D is the midpoint of side AB. It follows that triangles DAC and DBC have equal area, even though these two triangles are ***not*** congruent. Suppose that your teacher asks you to cut triangle DAC into pieces in such a way that the pieces may be reassembled to form triangle DBC. What is the smallest number of pieces needed? Explain your answer completely!