THE INCLUSION-EXCLUSION PRINCIPLE

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The inclusion-exclusion principle (like the pigeon-hole principle we studied last week) is simple to state and relatively easy to prove, and yet has rather spectacular applications. In class, for instance, we began with some examples that seemed hopelessly complicated. Shortly we will show they are really easy applications of the inclusion-exclusion principle.

Suppose A_1 and A_2 are any sets. Then it's easy to see that (by drawing a picture for instance) that

$$|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|.$$

A slightly more intricate picture shows (for any sets A_1, A_2, and A_3) that

$$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|.$$

So one might guess that for n sets $A_1, \ldots, A_n$,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1 \leq i_1 \leq n} |A_{i_1}| \; - \sum_{1 \leq i_1 < i_2 \leq n} |A_{i_1} \cap A_{i_2}| \tag{1}$$

$$+ \sum_{1 \leq i_1 < i_2 < i_3 \leq n} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| \; - \; \cdots \; + \; (-1)^{n-1} \sum_{1 \leq i_1 < \cdots < i_{n-1} \leq n} |A_{i_1} \cap \cdots \cap A_{i_{n-1}}|$$

$$+ (-1)^n |A_{i_1} \cap \cdots \cap A_{i_n}|.$$

This is indeed correct and is usually called the inclusion-exclusion principle.

How would one prove the general version (1)? Induction is one option. We already checked the case of $n = 2$. So assume (1) holds to give an expression for $|B|$ with $B = A_1 \cup \cdots \cup A_{n-1}$. Then, again by the $n = 2$ case,

$$|A_1 \cup \cdots \cup A_n| = |B \cup A_n| = |B| + |A_n| - |B \cap A_n|. \tag{2}$$

By induction, we have an expression for $|B|$. Using it, we must reduce (2) to (1). This isn't completely trivial, but it isn't very hard either. I'll leave it for you to do.

I promised spectacular applications. Here is one of them. Suppose n people are seated in a room with n chairs. They are all asked to get up and move to a different seat. Let $D(n)$ denote the number of possible new seating configurations; that is, $D(n)$ denotes the number of ways to rearrange n objects in such a way that no object is fixed by the rearrangement. (Such a rearrangement is sometimes called a *derangement*.) Now recall the number e. Like

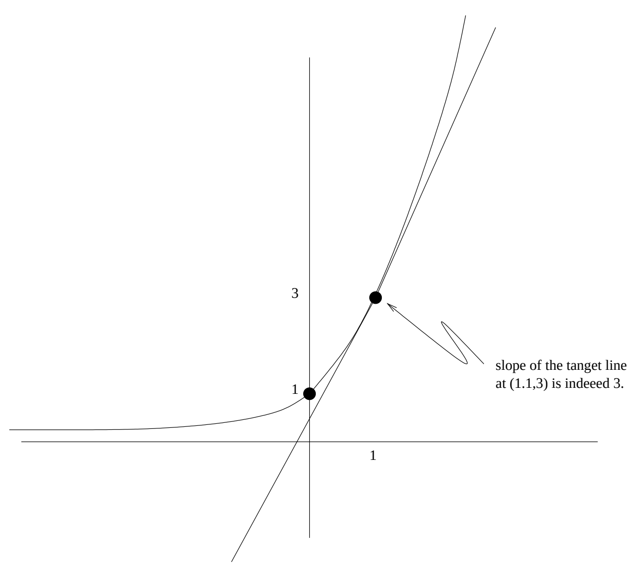
π, e is a very special number and we'll return its definition below. It is a remarkable fact that

$$\lim_{n\to\infty} \frac{D(n)}{n!} = \frac{1}{e}. \tag{3}$$

Some of you may know what limits are. For those of you that don't, the expression in (3) may informally be taken to mean "as n gets larger and larger, $D(n)/n!$ becomes a better and better (in fact, arbitrarily good) approximation to $1/e$."

Note why (3) is remarkable: the left-hand side involves only counting derangements, but the right-hand side is something (as we'll see below) that is irrational (and, in fact, transcendental). Strange, huh?

To make sense of (3), we must understand the number e. Yes, it's the same e that sits on the e^x button of your calculator. But how does one define e^x or, in particular, $e = e^1$ or $e^{-1} = 1/e$? This is actually rather tricky, at least in the sense that it requires some rather sophisticated ideas. Perhaps this is the reasons why most people who use their e^x button all the time don't know how it is actually defined! We start with a geometric definition. Draw a curve in the plane passing through $(0, 1)$ so that the slope of the tangent line to any point (x, y) on the curve is equal to y. You can quickly see that the curve must look something like



It transpires that there is a unique such curve, and that this curve is the graph of the function $f(x) = e^x$. This gives one way to define e^x. It is a nice intuitive definition, but it has some problems. The main one is making precise what one means by "the tangent line to the curve at (x, y)?" Another subtlety is that the curve so defined is actually unique. Both issues require some understanding of calculus, so we'll pursue another idea.

Define

$$e^x = 1 + x + \frac{x}{2!} + \frac{x}{3!} + \frac{x}{4!} + \cdots .$$

For instance,

$$e = e^1 = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \cdots \tag{4}$$

and

$$\frac{1}{e} = e^{-1} = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \cdots . \tag{5}$$

The subtlety here is making sense out of what one means by an infinite sum. Some of you may already know how to do this. If you don't, you can informally define the above infinite sum as follow: e^x is the number so that the sequence

$$1, 1+x,\ 1 + x + \frac{x}{2!},\ 1 + x + \frac{x}{2!} + \frac{x^2}{3!},\ \ldots$$

becomes a better and better (in fact, arbitrarily good) approximation to e^x. So this definition also requires a little sophistication.

As a quick aside, once we absorb (or accept) the definition of e given in (4), it is not hard to prove that e is irrational. (Remember that earlier we had proved that $\sqrt{2}$ is irrational.) Actually it's a little bit simpler to prove that $\frac{1}{e}$ is irrational. This of course immediately implies that e is also irrational. (Make sure you see this.) The key is that the definition immediately implies that

$$\begin{aligned}
\frac{1}{2!} - \frac{1}{3!} <& \frac{1}{e} < \frac{1}{2!} \\
\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} <& \frac{1}{e} < \frac{1}{2!} - \frac{1}{3!} \\
\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} <& \frac{1}{e} < \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} \\
& \vdots
\end{aligned}$$

and so on. (Again make sure that you see why this is true!) Now place the above expressions over a common denominator. For the first few we have

$$\begin{aligned}
\frac{2}{6} <& \frac{1}{e} < \frac{3}{6} \\
\frac{8}{24} <& \frac{1}{e} < \frac{9}{24} \\
\frac{44}{120} <& \frac{1}{e} < \frac{45}{120} \\
\frac{264}{6} <& \frac{1}{e} < \frac{265}{720} \\
& \vdots
\end{aligned}$$

and so on. The first equation implies that if $1/e$ were rational and written in lowest terms, its denominator could not divide $3! = 6$; the second says the denominator could not divide

$4! = 24$; the fourth says that the denominator could not divide $5! = 120$; and so on. The conclusion is that if $1/e$ were rational and written in lowest terms, say with denominator N, then N could not divide $N!$. But this is absurd. So this contradiction shows that $1/e$ and hence e is irrational.

Now back to (3). Since we have defined e (and $1/e$) precisely, in order to establish (3) we need to get a handle on the number $D(n)$. Here is where inclusion-exclusion enters. Let A_i be the number of rearrangements of n objects so that the ith object is fixed by the rearrangement. Then it is clear that $A_1 \cup A_2 \cup \cdots \cup A_n$ consists of all the rearrangements that leave *some* object fixed. We are interested in the rearrangements that leave *none* fixed, and since there are $n!$ total rearrangements, we conclude that

$$D(n) = n! - |A_1 \cup \cdots \cup A_n|.$$

The inclusion exclusion principle is designed to compute $|A_1 \cap \cdots \cap A_n|$. In order to do so, for any $1 \leq i_1 < \cdots < i_k \leq n$, we have to compute

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|.$$

The virtue of this approach is that $|A_{i_1} \cup A_{i_2} \cap \cdots \cup A_{i_k}|$ is indeed computable! In fact

$$A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}$$

is simply the set of permutations that fix $i_1, \ldots, i_k$, i.e. permuations of the remaining $n-k$ objects. So, indeed,

$$|A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}| = (n-k)!.$$

Now we apply (1) to conclude that

$$|A_1 \cup \cdots \cup A_n| = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)! = \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}.$$

Then it follows that

$$\begin{aligned} D(n) &= n! \ - \ |A_1 \cup \cdots \cup A_n| = n! \ - \ \sum_{k=0}^{n} (-1)^k \frac{n!}{k!} \\ &= n! \left(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^n \frac{1}{n!}\right). \end{aligned}$$

And so (3) is clear.

We'll now turn to the problems. I found many of these on the web by googling `inclusion exclusion problems`. By doing so, you can find many more.

EXERCISES

1. Among 18 students in a room, 7 study mathematics, 10 study science, and 10 study computer programming. Also, 3 study mathematics and science, 4 study mathematics and computer programming, and 5 study science and computer programming. We know that 1 student studies all three subjects. How many of these students study none of the three subjects?

2. Let A,B, and C be sets with the following properties:

- $|A| = 100$, $|B| = 50$, and $|C| = 48$.
- The number of elements that belong to exactly one of the three sets is twice the number that belong to exactly two of the sets.
- The number of elements that belong to exactly one of the three sets is three times the number that belong to all of the sets.

How many elements belong to all three sets?

3. Three sets A,B, and C have the following properties: $|A| = 63$, $|B| = 91$, $|C| = 44$, $|A \cap B| = 25$, $|A \cap C| = 23$, $|C \cap B| = 21$. Also, $|A \cup B \cup C| = 139$. What is $|A \cap B \cap C|$?

4. Two circles and a triangle are given in the plane. What is the largest number of points that can belong to at least two of the three figures?

5. Fix a regular hexagon. Let S denote its vertices, together with its center. (Draw a picture if the situation is confusion.) How many equilateral triangles have at least two vertices in S.

6. (a) How many integers between 1 and 2005 are NOT multiples of any of the numbers 2, 3 or 5?

(b) How many integers in the set $\{1, 2, 3, 4, ..., 360\}$ have at least one prime divisor in common with 360?

(c) Find the number of integers x such that $1 \leq x \leq 2004$ and x is relatively prime to 2005.

7. All the phone numbers in Nowheresville either start with 56, or end with 7, or both. Otherwise, the digits of the phone number can be any of the digits 0–9. How many possible phone numbers exist in Nowheresville?

7. Let $U = \{1, \ldots, 1000\}$ and define subsets A_2, A_3, A_5 as follows,

$$A_2 = \{n \mid 1 \leq n \leq 1000 \text{ and } n \text{ is even}\}$$

$$A_3 = \{n \mid 1 \leq n \leq 1000 \text{ and } n \text{ is a multiple of } 3\}$$

$$A_5 = \{n \mid 1 \leq n \leq 1000 \text{ and } n \text{ is a multiple of } 5\}$$

For each A_i, write $\bar{A}_i$ for $U \backslash A_i$ (the complement of A_i in U). Find the number of elements of each of the sets listed below

(a) $A_2 \cap A_3 \cap A_5$
(b) $A_2 \cap A_3 \cap \bar{A}_5$
(c) $A_2 \cap \bar{A}_3 \cap A_5$
(d) $\bar{A}_2 \cap A_3 \cap A_5$
(e) $\bar{A}_2 \cap \bar{A}_3 \cap A_5$
(f) $\bar{A}_2 \cap A_3 \cap \bar{A}_5$
(g) $A_2 \cap \bar{A}_3 \cap \bar{A}_5$
(h) $\bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_5$

8. In a math contest, three problems, A, B, and C were posed. Among the participants there were 25 who solved at least one problem. Of all the participants who did not solve problem A, the number who solved problem B was twice the number who solved C. The number who solved only problem A was one more than the number who solved A and at least one other problem. Of all participants who solved just one problem, half did not solve problem A. How many solved just problem B?

9. How many numbers can be obtained as the product of two or more of the numbers 3,4,4,5,5,6,7,7,7?

10. How many of the first 100 positive integers are expressible as a sum of three or fewer members of the set $\{1, 3, 9, 27, 81\}$ if we are allowed to use the same power more than once. For example, 5 can be represented, but 8 cannot.

11. How many integers can be expressed as a sum of two or more different members of the set 0,1,2,4,8,16,31?

13. Of 28 students taking at least one subject, the number taking Math and English but not History equals the number taking Math but not History or English. No student takes English only or History only, and six students take Math and History but not English. The number taking English and History but not Math is 5 times the number taking all three subjects. If the number taking all three subjects is even and non-zero, how many are taking English and Math but not History?

13. In a survey of the chewing gum tastes of a group of baseball players, it was found that:

> 22 liked juicy fruit;
> 25 liked spearmint;
> 39 like bubble gum;
> 9 like both spearmint and juicy fruit;
> 17 liked juicy fruit and bubble gum;
> 20 liked spearmint and bubble gum;
> 6 liked all three;

Given that four liked none of the above, how many baseball players were surveyed?

14. Mr. Brown raises chickens. Each can be described as thin or fat, brown or red, hen or rooster. Four are thin brown hens, 17 are hens, 14 are thin chickens, 4 are thin hens, 11 are thin brown chickens, 5 are brown hens, 3 are fat red roosters, 17 are thin or brown chickens. How many chickens does Mr. Brown have?

15. Consider the following information regarding three sets A, B, and C all of which are subsets of a set U. Suppose that $|A| = 14, |B| = 10, |A \cup B \cup C| = 24$ and $|A \cap B| = 6$. Consider the following assertions:

(1) C has at most 24 members
(2) C has at least 6 members
(3) $A \cup B$ has exactly 18 members

Which ones are true?

16. There are 15 students seated in classroom. The teacher is not satisfied with the seating arrangement and demands that everyone move to a new seat. How many new configurations are possible?

17. How many 10 digit phone numbers contain at least one of each odd digit?

18. At the annual Granite High Foxtrot-Til-You-Drop dance, 20 couples are foxtrotting peacefully. Of the 20 couples, 10 are jock-cheerleader couples. The principal arrives and decides that things are getting a little too steamy. He asks that everyone switch to a new partner. Of course the jocks again end up with the cheerleaders. Given this, how many new configurations are possible?

19. Michael has a new cell phone and he's having difficulty remembering the new ten-digit phone number. His memory is bizarrely fragmented: he remembers that that the second, fourth, and fifth digits are either 7 or 9, the third and tenth digits are either a 2 or 4, there are two zeros in the number, and the sum of the digits is 42. Given this information, how many possibilities are there for Michael's phone number?