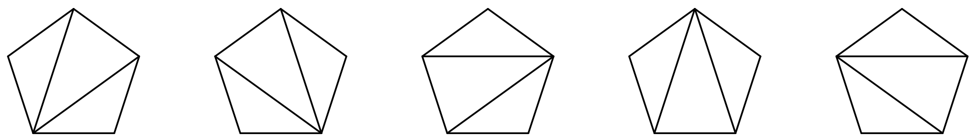
Catalan numbers

1. In how many ways can you cut up a regular hexagon (6-sided polygon) into triangles using 3 nonintersecting diagonals? For example, there are 5 ways for a pentagon:



2. In how many ways can you place 3 pairs of parentheses in the expression

$$1 + 2 + 3 + 4 + 5$$

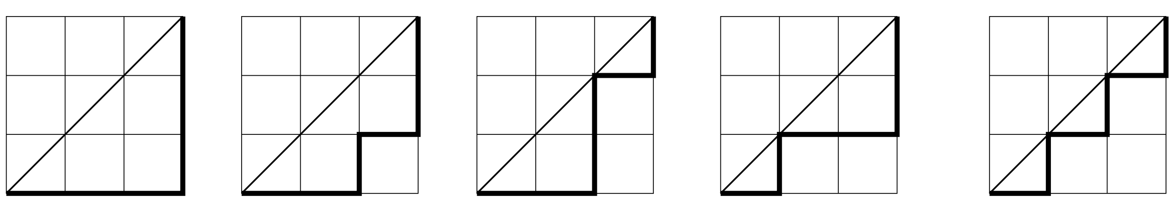
to specify the order of addition? For example, there are 5 ways for the expression $1 + 2 + 3 + 4$:

((1+2)+3)+4   1+(2+(3+4))   (1+(2+3))+4   (1+2)+(3+4)   1+((2+3)+4)

3. Alice and Bert are the two candidates in the Catalan local elections. In a closely fought contest, the result was a draw! Interestingly, the whole way through the count Alice never trailed Bert. If there were 8 votes in all, in how many ways could the count have gone? For example, if there were 6 votes, the possibilities would have been:

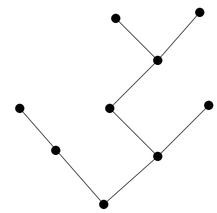
$$AAABBB \quad AABABB \quad AABBAB \quad ABAABB \quad ABABAB$$

4. The streets of Catalanville are arranged in a rectangular grid whose diagonal is on the shores of the Catalan Sea. In how many ways can one walk 4 blocks east and 4 blocks north starting and ending on the shoreline? For example, there are 5 possibilities for 3 blocks.



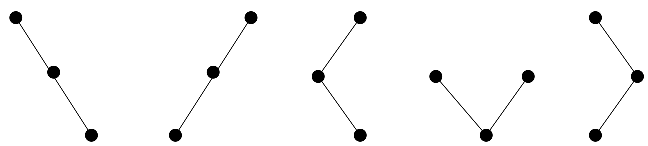
5. A *binary tree* is a graph that consists of nodes and branches. The rules are:

(1) There is a distinguished node called the *root*, drawn at the bottom.
(2) Each node either grows two branches, one on the left and one on the right, or it grows one branch (on the left or on the right), or it does not grow branches at all.
(3) Each node except the root are on top of exactly one branch.
(4) The root is not on top of any branches.



A 9-node binary tree

How many binary trees are there with 4 nodes? There are 5 binary trees with 3 nodes.



6. A *Conjecture* is a mathematical statement whose author suspects that it might be true. Before mathematicians prove a theorem they usually state a Conjecture and then they work on proving it.

Can you state a Conjecture after working out problems 1-5?

Part 2

We will now try to see if there is a better way to calculate the numbers from part 1, without having to draw all possibilities. We will start with the map of Catalanville, but as a warm-up we will first think about the same problem for an inland Catalan town by the name of Sandypolis.

7. The streets of Sandypolis are laid out in a grid. In how many ways can you walk m blocks east and n blocks north? Call this number $f(m,n)$. Compute $f(1,0), f(0,1), f(1,1), f(2,1), f(1,2), f(2,2)$. To keep track of these numbers, draw a grid and write the number $f(m,n)$ at the position (m,n) reached from $(0,0)$ by walking m blocks east and n blocks north.

What are the two possibilities for the next-to-last position for a path before reaching (m,n)? How many (in terms of the function f) of each kind of paths are there? Prove that the number in position (m,n) is equal to the sum of the numbers in positions $(m-1,n)$ and $(m,n-1)$. Now write these numbers in your grid for all (m,n) with $m+n \leq 10$.

Have you seen these numbers before? Look at the ones on the line $m+n = 2$, $m+n = 3$ etc. These numbers are called *binomial coefficients* and they are denoted $f(m,n) = \binom{m+n}{n}$. Do you know why they are called binomial coefficients? Compare with

$$\begin{aligned}(x+y)^2 &= x^2 + 2xy + y^2\\(x+y)^3 &= x^3 + 3x^2y + 3xy^2 + y^3\\(x+y)^4 &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

8. We are now ready for Catalanville! Thus only positions (m,n) with $m \geq n$ are valid (unless you want to get wet!) Denote by $g(m,n)$ the number of paths that go from $(0,0)$ to (m,n). What are the possibilities for the next-to-last position for a path before reaching (m,n)? Note that the answer depends on whether $m > n$ or $m = n$. Write down the equation that expresses $g(m,n)$ in terms of $g(m,n-1)$ and $g(m-1,n)$ analogous to the equation $f(m,n) = f(m-1,n)+f(m,n-1)$ that holds for binomial coefficients. Such an equation is called a *recursive formula.* It enables you to quickly calculate the values of the function. Compute $g(n,n)$ for $n = 1,2,3,\cdots,10$.

Part 3

We will now discuss the conjecture from Problem 6. We start with a bit of theory.

How does one compare the sizes of two finite sets? Of course, you can count them both. But sometimes that's too hard. Imagine you are in a movie theater watching a movie. Are there more seats or movie goers? I bet you know the answer without counting!

Let's now say this more formally. Recall that a function

$$f : A \to B$$

assigns an element $f(a)$ of the set B to each element a of A. For example, A could be the set of movie-goers, and B could be the set of seats. If we make the assumption that each movie-goer occupies one and only one seat[1] then we have a function $f : A \to B$ that to each movie-goer assigns the seat he/she occupies.

A function $f : A \to B$ is *injective* if $f(a_1) = f(a_2)$ implies that $a_1 = a_2$. For our function, this means that a seat is occupied by at

[1]legs spread out across another seat don't count

most one patron[2]. A function $f : A \to B$ is *surjective* if for every $b \in B$ there is $a \in A$ such that $f(a) = b$. Our function is surjective provided every seat is occupied. A function that is both injective and surjective is *bijective*.

Definition. Two sets A and B have the same cardinality[3] if there exists a bijective function $f : A \to B$.

Next, we describe four sets.

Triangulations. A *triangulation* of a polygon is a way of cutting it up into triangles using nonintersecting diagonals.

$$TRIANG(n)$$

is the set of triangulations of the regular $(n+2)$-gon[4]. For example, Problem 1 lists all 5 elements of $TRIANG(3)$ and you worked out that $TRIANG(4)$ has 14 elements.

Catalanville paths.

$$CATGRIDPATH(n)$$

is the set of all paths of length $2n$ in a grid joining points $(0,0)$ and (n,n) and never crossing above the diagonal $y = x$. For example, $CATGRIDPATH(3)$ has 5 elements and you found that $CATGRIDPATH(4)$ has 14 elements.

Parentheses.

$$PARENS(n)$$

is the set of ways of placing n pairs of parentheses in the expression

$$1 + 2 + \cdots + n + (n+1)$$

to determine the order of addition. For example, $PARENS(3)$ has 5 elements and you found that $PARENS(4)$ has 14 elements.

[2] no babes in arms

[3] "cardinality" means "number of elements"

[4] It may seem funny that we are looking at $(n+2)$-gons and not n-gons. How many triangles does a triangulation of an $(n+2)$-gon have?

Binary trees.

$$BINTREES(n)$$

is the set of binary trees with n nodes. For example, $BINTREES(3)$ has 5 elements and you found that $BINTREES(4)$ has 14 elements.

Big Conjecture. For any n the sets
$TRIANG(n)$, $CATGRIDPATH(n)$, $PARENS(n)$, $BINTREES(n)$
have the same cardinality.

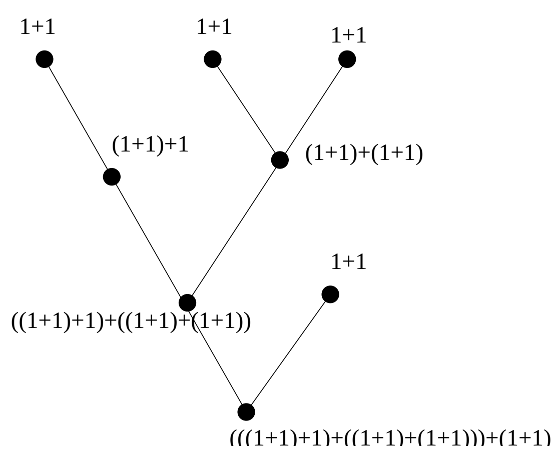
We know the BC holds for $n = 3$ and (thanks to your homework) for $n = 4$. The cases $n = 1, 2$ are easy and left to the reader[5].

Next, we should try to define functions between these sets that have a chance of being bijective. I will draw some picture to suggest the functions.

$PARENS(n)$ **vs.** $BINTREES(n)$**.** It is somewhat more convenient to think of the expression

$$1 + 1 + \cdots + 1$$

($n + 1$ 1's).



9. Find a consistent rule for associating a binary tree to a parenthesized expression. Conversely, find a rule that to a binary tree associates an expression. Mathematically, construct functions

$$F : PARENS(n) \to BINTREES(n) \text{ and } G : BINTREES(n) \to PARENS(n)$$

[5]This phrase is very common in math papers.

and verify that both are bijective functions. In fact, show that $F \circ G = identity$, $G \circ F = identity$. Thus, you proved the

Theorem. *$PARENS(n)$ and $BINTREES(n)$ have the same cardinality.*

Note that we still don't know how big are $PARENS(15)$ and $BINTREES(15)$, but we know they have the same number of elements! Just like movie-goers and seats.

Parentheses to paths. .

10. Show that $CATGRIDPATH(n)$ has the same cardinality as the set of sequences consisting of n A's and n B's so that the A-count never trails the B-count. (Hint: A=move right, B=move up)

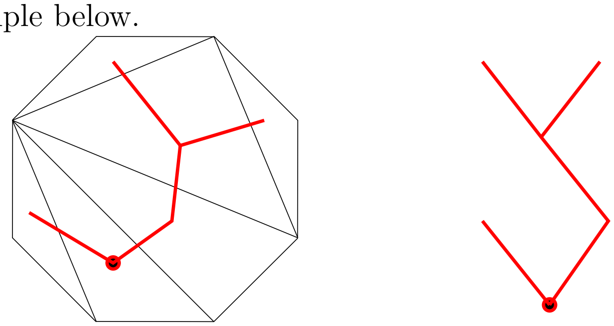
11. Define a function $H : PARENS(n) \to CATGRIDPATH(n)$ using the following as a model.

$$((1{+}1){+}1){+}(1{+}1) \to (((1{+}1){+}1){+}(1{+}1)) \to (((+{+}+(+ \to AAABBBAB$$

12. Prove that H is surjective. In other words, if someone hands you a string of ('s and +'s obtained by erasing 1's and)'s from an expression, can you reconstruct the expression? Also prove that H is injective.

Triangulations to binary trees. .

13. Define a function $I : TRIANG(n) \to BINTREES(n)$ modeled on the example below.



14. This one is a real challenge! Can you define a function

$$J : CATGRIDPATH(n) \to BINTREES(n)$$

that has a reasonable chance of being bijective?

15. This one is even harder. Finish the proof of the Big Conjecture.

Definition. The common cardinality of the four sets is called the n^{th} Catalan number $C(n)$.

Problem 7 gives you a way to calculate these numbers recursively. Here is what I got:

n	1	2	3	4	5	6	7	8	9	10	11
C(n)	1	2	5	14	42	132	429	1430	4862	16796	58786

Part 4: A closed formula

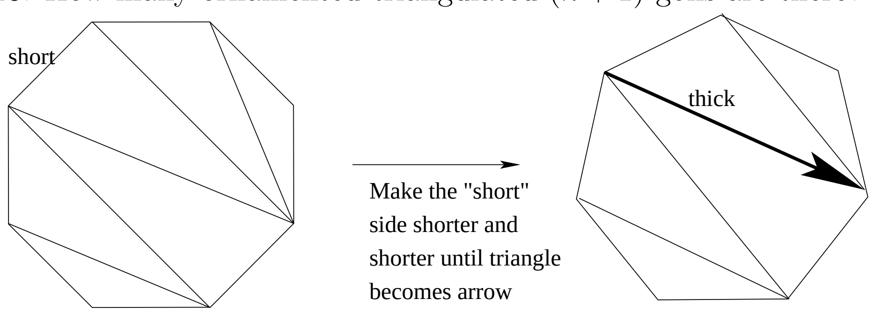
By a "decorated triangulated polygon" we will mean a triangulated polygon with one of the non-bottom sides labeled "short".

16. Draw pictures of some decorated triangulated polygons. How many decorated triangulated $(n+2)$-gons are there? Your answer will involve $C(n)$.

17. When you triangulate an $(n+1)$-gon, how many diagonals do you draw?

By an "ornamented triangulated polygon"[6] we will mean a triangulated polygon with one of the sides or one of the diagonals labeled "thick" and oriented by an arrow in one of two possible directions.

18. How many ornamented triangulated $(n+1)$-gons are there?



19. Prove that $(n+1)C(n) = 2(2n-1)C(n-1)$.

20. Prove that

$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$

21. Prove geometrically the following recursive relation:

$$C(n) = C(0)C(n-1) + C(1)C(n-2) + C(2)C(n-3) + \cdots + C(n-1)C(0)$$

where we take $C(0) = 1$. Hint: How many triangulations of an $(n+2)$-gon are there where the bottom side forms a triangle with a particular vertex?

[6]When it comes to naming things, mathematicians are extremely unimaginative.

Part 5

The following two problems are more difficult. I am offering a free pizza for the solution of the second one (I don't know how to do it).

22. (International Math Olympiad 1996, Mumbai, India) Let p, q, n be three positive integers with $p + q < n$. Let $(x_0, x_1, \ldots, x_n)$ be an $(n+1)$-tuple of integers satisfying the following conditions:

(a) $x_0 = x_n = 0$.

(b) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

23. Find a closed formula for $g(m, n)$ from problem 8.

Web resources:

http://www-groups.dcs.st-andrews.ac.uk/~history/Miscellaneous/CatalanNumbers/catalan.html

http://www.saintanns.k12.ny.us/depart/math/Seth/toc.html

http://www.maths.usyd.edu.au:8000/u/don/code/Catalan/Catalan.html